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HISTORY

Bruins, E. M. Ancient Egyptian arithmetic: 2/*N*. Nederl. Akad. Wetensch. Proc. Ser. A. 55 = Indagationes Math. 14, 81–91 (1952).

The question how the $(2/n)$-table in the Papyrus Rhind was constructed by the ancient Egyptian mathematicians has been answered in different ways by O. Neugebauer and B. L. van der Waerden. The author of the present paper rejects both explanations and proposes a new one. According to this a series of decompositions for prime numbers can be obtained by a very simple purely additive process. The extant deviations from this scheme follow from some general principles which clearly underlie the construction. The trivial decomposition for 101, which was considered by Neugebauer as an addition by an unskilled copyist, is proved to be the only possible one in accordance to one of these principles.

E. J. Dijksterhuis (Oisterwijk).

***Arithmetik des Diophantos aus Alexandria.** Aus dem Griechischen übertragen und erklärt von Arthur Czwalina. Abh. Math. Sem. Univ. Hamburg, Beiheft 1. Vandenhoeck & Ruprecht, Göttingen, 1952. 148 pp. 20 DM.

This is a translation into German of Tannery's Greek text of the extant mathematical works of Diophantus [Teubner, Leipzig, 2 vols., 1893, 1895]. Complete information concerning the text editions and the translations up to 1927 is given by Sarton [Introduction to the history of science, vol. I, Williams and Wilkins, Baltimore, 1927, pp. 336–337]. Specimens of the Greek text with English translations may be found in "Selections illustrating the history of Greek mathematics," vol. II [translated by Ivor Thomas, Loeb Classical Library, 1941, rev. ed., 1951; these Rev. 13, 419]. The author has furnished mathematical notes and explanations to most of the examples of the text.

E. B. Allen (Troy, N. Y.).

***Schriften des Nikolaus von Cues. Die mathematischen Schriften.** Übersetzt von Josepha Hofmann mit einer Einführung und Anmerkungen versehen von Joseph Ehrenfried Hofmann. Verlag von Felix Meiner, Hamburg, 1952. lii+268 pp.

The last edition of Cusanus' mathematical works appeared at Basle in 1565. This new edition, in German translation, is part of the planned Opera Omnia of Cusanus, in twenty volumes of the original Latin text and as many volumes as necessary of the German translation. Of this edition several volumes have appeared, the first in 1932. The twelve essays published and discussed in the present volume all deal with the rectification and the quadrature of the circle. They date from 1445 to 1459 and represent the testing on a purely mathematical problem of the general ideas concerning the infinite which Cusanus expressed in his philosophical works such as "De Docta Ignorantia" (published in another volume of the Opera Omnia). Of these twelve essays nine were published in the 1565 edition, three have so far been available only in manuscript. For the preparation of this edition all available manuscripts and printed texts have been carefully compared. Added are an introduction of 45 pages and an additional 65 pages of notes which allow us to follow the labors of the cardinal in modern notation and in relation to his contemporaries. Thus we owe to the conscientious scholarship of the editor and translator a valuable addition to our mathematical source material.

D. J. Struik (Cambridge, Mass.).

***Ten Doesschate, G. Rolduc als middeleeuwsche voorpost der wis-, natuur- en geneeskunde in de Nederlanden. [Rolduc as medieval vanguard of mathematics, physics and medicine in the Netherlands.]** De Tijdstroom, Lochem, 1948. 154 pp.

The catalogue of books in the possession of the ancient abbey Kloosterrade at Rolduc, probably dating from the first half of the thirteenth century, contains certain titles of books on arithmetic, astronomy, physics, medicine and biology. In this book an account is given of the content of these books. Special attention is given to Boethius' Arithmetica and the probably spurious Geometria (both from the Opera Omnia of 1570), to the calculus with the abacus, to the computus Paschalis and to the astrolabe.

D. J. Struik (Cambridge, Mass.).

***Newton, Sir Isaac. Opticks, or a treatise of the reflections, refractions, inflections and colours of light.** Dover Publications, Inc., New York, N. Y., 1952. cxv+406 pp. Paperbound, $1.90; clothbound, $3.95.

This edition of Newton's Opticks is based upon the 4th edition [London, 1730]. It contains additionally a preface by I. B. Cohen (pp. ix–lviii), a forward by A. Einstein (pp. lix–lx), an introduction by E. T. Whittaker (pp. lxi–lxxvii), and an analytical table of contents by D. H. D. Roller (pp. lxxix–cx). The preface and table of contents are new for this edition, the rest is reproduced by photo-offset from the edition of G. Bell & Sons, London, 1931.

***N'yuton, Isaak. Lekcii po optike. [Newton, Isaac. Lectiones opticae.]** Translation, commentary and editing by S. I. Vavilov. Izdat. Akad. Nauk SSSR. Moscow-Leningrad, 1946. 295 pp. (2 plates). 20 rubles.

A translation into Russian of Newton's "Lectiones opticae" [London, 1729]. The English translation of Part I, "Optical lectures" [London, 1728], was collated with the Latin text in translating this part.

***Whittaker, Edmund. A history of the theories of aether and electricity. Vol. I. The classical theories.** Philosophical Library, New York, N. Y., 1951. xiv+434 pp. $12.00.

This book is a second, much changed edition of one that appeared in 1910 under the title "A history of the theories of aether and electricity from the age of Descartes to the

close of the nineteenth century" [Longmans Green, London, 1910]. It is intended to be only the first of two volumes, the second of which will deal with the later development, starting with quantum and relativity theory and leading up to the problems of electrodynamics in the present day. The author claims that the title "Aether and electricity" is apt to describe even the latest development in our century. Indeed modern physics deals with the properties of the vacuum, and the author thinks that "it is absurd to retain the name 'vacuum' for an entity so rich in physical properties, and the historical word 'aether' may fitly be retained." The book begins with a chapter on "The theory of the aether to the death of Newton," and concludes with a chapter on "Classical theory in the age of Lorenz." Throughout the book, a unified notation is used (e.g., the vector notation is introduced as early as the chapter "From Galvani to Ohm"), which makes the entire development clear and understandable for the modern reader. This narrative faithfully records the slow and painful progress made in the development of the field concepts. Younger physicists especially will benefit from this chronicle of achievements which might otherwise be forgotten in the rapid contemporary progress in this field.

L. Infeld (Warsaw).

Severi, Francesco. I fondamenti remoti e prossimi della geometria algebrica. Univ. e Politecnico Torino. Rend. Sem. Mat. **10**, 67–95 (1951).

An historical sketch of the origins of algebraic geometry, followed by a survey of the contributions of the Italian geometers to the development of rigorous foundations of modern algebraic geometry. Within the self-imposed limits of the author's exposition no room was left for a brief mention of some more recent work on the foundations of algebraic geometry, done mainly by American and French mathematicians. *O. Zariski* (Cambridge, Mass.).

O*Markuševič, A. I. **Očerki po istorii teorii analitičeskih funkcii. [Essays on the history of the theory of analytic functions.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 127 pp. 4.30 rubles.

This is admittedly not intended as a connected or complete account of the development of the theory of functions of a complex variable, but rather as a series of independent historical essays on topics of interest to Russian readers. The first essay is entitled "The collection of fundamental facts in the eighteenth century," and deals in detail with what W. F. Osgood used to call the age of fable. The author stresses Euler's contributions. The second essay, "The development of a systematic theory of analytic functions," carries the story in equal detail up to the end of the nineteenth century, though there are some curious omissions (for example, there is no reference to any proof of the Riemann mapping theorem except Hilbert's justification of Dirichlet's principle). Russian workers receive only the attention warranted by their achievements or their special interest to Russian readers. The third essay is a short account of the use of Lobachevskian geometry in the theory of functions. Essay four, "The ideas of P. L. Čebyšev on the approximation of functions and their development in the theory of analytic functions," deals with various topics in what is sometimes called the constructive theory of functions. Essay five is "The work of Soviet mathematicians in the theory of analytic functions, in connection with problems of mechanics, the theory of functions of real variables and the theory of numbers." There are a few references to foreign mathematicians in this essay but (as the title suggests) no attempt is made to correlate Russian and foreign work. The author seems to have consulted many primary sources and his statements are usually accurate (though he interchanges the nationalities of Heaviside and Porter). One might wish that he had written a more inclusive work, but within his self-imposed limits one cannot quarrel with his exposition. Nonmathematical topics are not mentioned.

R. P. Boas, Jr. (Evanston, Ill.).

*Bernštein, S. N. **Sobranie sočinenii. Tom I. Konstruktivnaya teoriya funkcii [1905–1930]. [Collected works. Vol. I. The constructive theory of functions [1905–1930].]** Izdat. Akad. Nauk SSSR, Moscow, 1952. 581 pp. (1 plate). 31.50 rubles.

This volume collects the author's papers (for the indicated period) on polynomials, trigonometric series, theory of approximation, interpolation, absolutely monotonic functions, entire functions, etc. Papers originally published in other languages are translated into Russian. There are 39 pages of notes by the author and a list of all his publications.

R. P. Boas, Jr. (Evanston, Ill.).

*Freudenthal, Hans. **Inleiding tot het denken van Einstein. [Introduction to the thinking of Einstein.]** Uitgeverij Born N.V., Assen, 1952. 56 pp. (1 plate). 1.45 florins.

One of a series of popular books, "Leading Figures in Human Thought".

Smirnov, V. I., and Bermant, A. F. Obituary: Gennadiĭ Mihaĭlovič Goluzin. Uspehi Matem. Nauk (N.S.) **7**, no. 3(49), 97–102 (1 plate) (1952). (Russian)

A discussion of Goluzin's mathematical work and a list of his published mathematical papers.

Freire, Luís. Concerning Gomes Teixeira. Gaz. Mat., Lisboa **12**, no. 50, 109–111 (1951). (Portuguese)

O*Luzin, N. N. **Integral i trigonometričeskiĭ ryad. Redakciya i kommentarii N. K. Bari i D. E. Men'šova. [Integral and trigonometric series. Editing and commentary by N. K. Bari and D. E. Men'šov.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 550 pp. (1 plate). 20.45 rubles.

This edition of Luzin's dissertation [Moscow University, 1916] (pp. 49–386) is accompanied by a biography (pp. 11–31) by V. V. Golubev and N. K. Bari, an essay, "On N. N. Luzin's book, 'Integral and trigonometric series', and his work in the metric theory of functions" (pp. 32–45) by N. K. Bari and L. A. Lyusternik, a commentary (pp. 389–537) by N. K. Bari and D. E. Men'šov, and a list of Luzin's published works.

Bari, N. K., and Lyusternik, L. A. Dmitriĭ Evgen'evič Men'šov (for his sixtieth birthday). Uspehi Matem. Nauk (N.S.) **7**, no. 3(49), 145–150 (1 plate) (1952). (Russian)

A list of Men'šov's published papers is included.

FOUNDATIONS

✓ ✱**Mostowski, Andrzej. Sentences undecidable in formalized arithmetic. An exposition of the theory of Kurt Gödel.** Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1952. viii+117 pp. 12 florins. $3.25.

This is an up-to-date exposition of Gödel's theory on the incompleteness of functional calculi which include ordinary arithmetic. In spite of the author's modest remarks, no strictly comparable work exists, at least to the reviewer's knowledge. There are indeed other books which contain chapters on Gödel's works, but none is entirely focussed on it like the present monograph. Apart from that, the author makes an original contribution to the subject by providing a unified treatment of the various methods of approach which have been put forward.

The contents of the book are as follows. The introduction contains a largely informal discussion of the notion of incompleteness and a sketch of the fundamental idea of Gödel's argument. Following certain auxiliary arithmetical theorems and recursive definitions, the object language (S) is introduced and arithmetised. (S) and its syntax are then investigated in detail. There follows a discussion of the semantics of (S), based on Tarski's work. To achieve a unified treatment of the subject the author then introduces the notion of definability with respect to any given class of statements which is consistent and deductively closed. This includes Tarski's notion of definability and the notion of general recursivity as special cases. Gödel's main theorem is given in the last chapter with the extensions and corollaries of Rosser and Tarski. Gödel's "second incompleteness theorem" is given in the appendix. There follows a discussion of the problem, first reviewed by Gödel and considered in more detail by Tarski and others, how to extend the incomplete calculus of (S) in order to derive the theorem which was previously unprovable. [Reviewer's comment. The addition of supplementary rules ad hoc—and indefinitely—does not solve the problem whether there is a criterion which would replace the "intuitive" opinion of all mathematicians, who are not intuitionists, that a given type of mathematical argument is acceptable. The fact is that such almost universal agreement exists for all mathematical theories developed so far.] The book concludes with a section based on Gödel's ideas on the length of proofs. *A. Robinson.*

✱**Becker, Oskar. Untersuchungen über den Modalkalkül.** Westkulturverlag Anton Hain, Meisenheim/Glan, 1952. 87 pp. DM 14.50.

In the introductory chapter of this work the author develops a reduced form of Lewis's system of strict implication (S_2). He adds to the primitives of the classical propositional calculus the primitive N denoting necessity and formalizes the resulting system by adding to the axioms of the classical propositional calculus the two further axioms: I) $N(p, q) \equiv Np.Nq$; II) $Np \supset p$. In addition to the rules of procedure of the classical propositional calculus he uses the following rule: If the two expressions A^0 and B^0 contain no modal symbols, then $A^0 \equiv B^0 \rightarrow NA^0 \equiv NB^0$. He then develops a similar system based on strict implication and compares these systems with S_2.

In the first chapter a statistical interpretation of the modal calculus is given and first-order modalities are interpreted in terms of the first-order predicate calculus. Methods are developed for comparing the relative "strengths" of modalities. The second chapter deals with normative interpretations of the modal calculus. The modalities are interpreted in terms of metamathematical concepts and also in terms of juridical concepts.

In the third chapter modalities are introduced into the predicate calculus and an application is made to the problem of the "Tertium non datur" for infinite object-regions. In the fourth chapter the author discusses different meanings which can be attached to the modalities in various systems. The book concludes with a number of appendices. The author proves the independence of his axioms I and II and discusses the possibility of interpreting modal logic as a many-valued logic. He then proves the consistency of one of his methods for comparing the strengths of modalities and further discusses his additional rule of procedure.

Erratum: Abb. 5 (p. 27) should be interchanged with Abb. 13 (p. 30). *A. Rose* (Nottingham).

Rose, Alan. Sur un ensemble de fonctions primitives pour le calcul des prédicats du premier ordre, lequel constitue son propre dual. C. R. Acad. Sci. Paris **234**, 1830–1831 (1952).

Die Konstanten 0, 1, und $[A, B, C, A] = (A \wedge B) \vee (C \wedge \bar{A})$ bilden eine selbstduale unabhängige Basis der Wahrheitsfunktionen des Aussagenkalküls [A. Church, Portugaliae Math. **7**, 87–90 (1948); diese Rev. **10**, 421]. Verf. fügt $\{x\}[A, B(x), C(x), A]\{x\} = [A, \wedge_x B(x), \vee_x C(x), A]$ hinzu und erhält eine solche Basis im Prädikatenkalkül. Analog für mehrere Wahrheitswerte. *P. Lorenzen* (Bonn).

Rose, Alan. Extensions of some theorems of Schmidt and McKinsey. I. Norsk Mat. Tidsskr. **34**, 1–9 (1952).

Die Systeme $(p_1, \cdots, p_{m-1})$ mit $p_1 \leqq \cdots \leqq p_{m-1}$ aus einem Booleschen Verband bilden eine Algebra, die zur Untersuchung der m-wertigen Logik dient. Für Boolesche Funktionen dieser Algebra werden einige Sätze beweisen, die für $m=2$ von J. C. C. McKinsey [Trans. Amer. Math. Soc. **40**, 343–362 (1936)] aufgestellt sind. *P. Lorenzen* (Bonn).

Rose, Alan. Some generalized Sheffer functions. Proc. Cambridge Philos. Soc. **48**, 369–373 (1952).

Verf. zeigt für die mehrwertigen Kalküle von Łukasiewicz, dass sich alle Wahrheitsfunktionen definieren lassen mit Hilfe einer einzigen. Für $m=\aleph_0$ muss vorher die Äquivalenz $\equiv$ zu den Funktoren hinzugenommen werden.
P. Lorenzen (Bonn).

Wang, Hao. Logic of many-sorted theories. J. Symbolic Logic **17**, 105–116 (1952).

Corresponding to a many-sorted theory T_n with n kinds of variables, a one-sorted theory $T_1^{(n)}$ can be set up. Schmidt [Math. Ann. **115**, 485–506 (1938)] has shown that a statement of any system T_n is provable in T_n if and only if its translation in the corresponding system $T_1^{(n)}$ is provable in $T_1^{(n)}$. He has further shown that given a statement of T_n and a proof for it in T_n, there is an effective way of finding a proof in $T_1^{(n)}$ for its translation in $T_1^{(n)}$; and, conversely, given a statement in $T_1^{(n)}$ which has a translation in T_n, and given a proof for it in $T_1^{(n)}$, there is an effective way of finding a proof in T_n for its translation in T_n. The author gives new proofs of the above theorems of Schmidt.

The author then develops a one-sorted theory Q corresponding to Gödel's system P [Monatsh. Math. Physik **38**,

173–198 (1931)], and gives an informal proof that if P is consistent, then Q is also consistent. *A. Rose.*

Curry, Haskell B. La théorie des combinateurs. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **10**, 347–359 (1951).

Einführung in die kombinatorische Logik als einer Methode die gebundenen Variablen aus der Mathematik zu eliminieren. *P. Lorenzen* (Bonn).

Curry, Haskell B. La logique combinatoire et les antinomies. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **10**, 360–370 (1951).

Es wird die Unverträglichkeit der kombinatorischen und deduktiven Vollständigkeit gezeigt. Anschliessend wird der Ansatz des Verf. mit Hilfe einer Theorie der Kategorien zu einem Verständnis der Antinomien zu kommen dargestellt. *P. Lorenzen* (Bonn).

Robinson, Julia. Existential definability in arithmetic. Trans. Amer. Math. Soc. **72**, 437–449 (1952).

Jede Relation ρ von natürlichen Zahlen, für die $\rho(x_1, \cdots, x_n)$ äquivalent ist einer Aussagenform, die aus 1, $=$, $+$, $\cdot$, und den logischen Konstanten $\wedge$, $\vee$, $\bigvee_x$ (also ohne $\bigwedge_x$, $\neg$) besteht, heisst "existenziell definierbar" (e.d.). Für jedes solche ρ gibt es ein Polynom P mit ganzen Koeffizienten, so dass

$$\rho(x_1, \cdots, x_n) \leftrightarrow \bigvee_{u_1, \cdots, u_k} P(x_1, \cdots, x_n; u_1, \cdots, u_k).$$

Es ist unbekannt, ob Pow—definiert durch x Pow $y \leftrightarrow \bigvee_z xz = y^2$—e.d. ist. Verf. beweist, dass Pow e.d. wäre, wenn es e.d. 2-stellige Relationen einer gewissen Wachstumsklasse gäbe. Wäre Pow e.d., dann wären auch die durch $x = y^2$, $x = y!$, x ist prim, dargestellten Relationen e.d. *P. Lorenzen.*

Łukasiewicz, Jan. On the intuitionistic theory of deduction. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 202–212 (1952).

Let I denote the intuitionistic calculus of propositions with constants F, T, O, N as implication, conjunction, disjunction, and negation. The system $I(TN)$ of the TN-theses of I is the same as the system $Cl(TN)$ of those of classical logic Cl; as Gödel remarked [Erg. Math. Kolloq. **4**, 34–38 (1933)], it follows that if we define C and A by T and N in the usual way, then Cl will be contained in I. Here it is shown that Cl is even contained in an axiomatized proper subsystem of $I(NT)$, if the classical conjunction Kpq is defined by $NNTpNNq$. The classical functors C, K, A all represent weaker relations than the corresponding F, T, O, as follows from known properties of I. *A. Heyting.*

Griss, G. F. C. Negationless intuitionistic mathematics. IVa. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 452–462 (1951).

This paper continues the author's outline of foundations of what he has called negationless mathematics [same Proc. **54** = Indagationes Math. **13**, 193–199 (1951); these Rev. **13**, 310]. In the absence of an adequate description of negationless logic, this work serves as a further indication of the author's intentions in this direction. Here he discusses an analytic geometry of the plane and derives theorems essentially equivalent to the axioms of Hilbert (the concept of rigid motion in the plane is taken as basic rather than that of congruence) [Grundlagen der Geometrie, 6th ed., Teubner, Leipzig-Berlin, 1923, Anhang II]. A point of the plane is defined as a nested sequence of λ_ν squares, where a λ_ν square is defined as a quadruple of pairs $(u,2^{-\nu}, v,2^{-\nu})$, $(u,2^{-\nu}+2^{1-\nu}, v,2^{-\nu})$, $(u,2^{-\nu}, v,2^{-\nu}+2^{1-\nu})$, $(u,2^{-\nu}+2^{1-\nu}, v,2^{-\nu}+2^{1-\nu})$. The author develops in separate sections, axioms for an affine plane geometry, axioms for the order relation, and axioms for motions in the plane. *D. Nelson* (Washington, D. C.).

Griss, G. F. C. Negationless intuitionistic mathematics. IVb. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 463–471 (1951).

A continuation of the work reviewed above. In particular, a distinctness relation for triangles is defined. Two triangles are said to be "discordant" just in case every motion carries one into a triangle different from the other. Examples are given of pairs of triangles such that it is impossible to prove one congruent to the other, but which are such that any triangle discordant with one is also discordant with the other. *D. Nelson* (Washington, D. C.).

Sampei, Yoemon. On the orthogonal expansion of the Boolean polynomial and its applications. I. J. Fac. Sci. Hokkaido Univ. Ser. I. **11**, 113–125 (1950).

I. R sei eine Menge, $x \varepsilon R$, $B = \{+1, -1\}$. Für $f \varepsilon \phi = B^R$ heisst $f_x = +1$ "primitive Aussage". Durch Negation, (verallgemeinerte) Konjunktion, und Disjunktion entstehen hieraus die "Aussagen" $P \varepsilon \mathfrak{P}$. P bzw. $\neg P$ wird bezeichnet durch P^{+1} bzw. P^{-1}. Jeder $F \varepsilon \mathfrak{F} = \mathfrak{P}^R$ heisst "Aussagenfunktion". $\mathfrak{A} \varepsilon \mathfrak{F} = \mathfrak{P} \mathfrak{F}$ heisst "Wahrheitsfunktion", wenn $F_1 \equiv F_2 \rightarrow \mathfrak{A}_{P_1} \equiv \mathfrak{A}_{P_2}$. $f^+ \varepsilon \mathfrak{F}$ wird definiert durch $f_x^+ \equiv (f_x = +1)$, $A \varepsilon \mathfrak{P}^\phi$ durch $A_f \equiv \mathfrak{A}_{f^+}$. Dann gilt:

$$\mathfrak{A}_P \equiv \bigvee_{A_f} \bigwedge_x F_x^{f_x} \equiv \bigwedge_{A_f^{-1}} \bigvee_x F_x^{-f_x} \quad \text{für alle } F.$$

Diese Normaldarstellungen von $\mathfrak{A}$ heissen "orthogonal expansions".

II. Mit ihrer Hilfe wird gezeigt dass für endlich viele Individuen die Axiome des Prädikatenkalküls die Quantoren als endliche Konjunktion bzw. Disjunktion charakterisieren. *P. Lorenzen* (Bonn).

***Carnap, Rudolf. The continuum of inductive methods.** The University of Chicago Press, Chicago, Ill., 1952. v+92 pp. $3.50.

This is a discussion of a special kind of confirmation functions $\mathfrak{c}(h, e)$ on a $\mathfrak{L}_N^\pi$ language. Let $a_1, \cdots, a_N$ be N individual constants and $P_1, \cdots, P_\pi$ primitive predicates (properties). The language formed by means of these signs and logical signs only is denoted by $\mathfrak{L}_N^\pi$. By Q-properties are designated those properties which are conjunctions of primitive properties, each primitive property occurring either unnegated or negated (e.g., $P_1 . \sim P_2 . P_3$ for $\pi = 3$). There are $\kappa = 2^\pi$ Q-properties. Each property of an individual expressible in $\mathfrak{L}_N^\pi$ may be uniquely represented as disjunction of Q-properties. The number w of Q-properties in this representation is called the logical width of this property. $w/\kappa = w/2^\pi$ is called the relative width. A state-description is a conjunction of the form $Q_{i_1}(a_1) \cdots Q_{i_N}(a_N)$ where the Q_i are Q-properties. There are κ^N state-descriptions. Each sentence in $\mathfrak{L}_N^\pi$ may be uniquely represented as disjunction of state-descriptions. If e_Q is a conjunction of the form $Q_{i_1}(a_{j_1}) \cdots Q_{i_s}(a_{j_s})$ and h_i a sentence $Q_i(a_{j_{s+1}})$ (each j_l different from all others), then e_i denotes a conjunction which is formed from e_Q by replacing every member $Q_{i_l}(a_{j_l})$, $i_l \neq i$, by $\sim Q_i(a_{j_l})$. The number of members for which $i_l = l$ (these members remain unchanged) is denoted by s_i.

The first part of this paper is concerned with the functions $\mathfrak{c}(h, e)$, where h and e are sentences of $\mathfrak{L}_N^\tau$ (e not logically false), which fulfil eleven conditions (C1–C11). The first five conditions are the usual conditions of confirmation functions (invariance with respect to logical equivalence, the multiplication and addition principles, and $0 \leqq \mathfrak{c}(h, e) \leqq 1$). C6. The function $\mathfrak{c}$ in $\mathfrak{L}_{N+1}^\tau$ is an extension of $\mathfrak{c}$ in $\mathfrak{L}_N^\tau$ (by definition $\mathfrak{L}_N^\tau$ is contained in $\mathfrak{L}_{N+1}^\tau$). C7, C8. $\mathfrak{c}(h_i, e_Q)$ is invariant with respect to permutations of Q-predicates, and individual constants. C9. $\mathfrak{c}(h_i, e_Q) = \mathfrak{c}(h_i, e_i)$. C10. $\mathfrak{c}(h_i, e_i)$ lies between s_i/s and $1/\kappa$ ($1/\kappa$ is the relative width of Q_i). C11. $[s.\mathfrak{c}(h_i, e_i) - s_i]/[1/\kappa - \mathfrak{c}(h_i, e_i)]$ depends on κ only and not on s, s_i. The author shows that: the values $\mathfrak{c}(h_i, e_i)$ determine all values of $\mathfrak{c}$; the values $\mathfrak{c}(h_i, e_i)$ depend on κ, s, s_i only. Thus the function $G(\kappa, s, s_i) = \mathfrak{c}(h_i, e_i)$ gives a complete characterization of $\mathfrak{c}$. In view of C10, $G(\kappa, s, s_i)$ may be expressed as $(s_i + \lambda/\kappa)/(s + \lambda)$, and from C11 it follows that λ depends on κ only: $\lambda = \lambda(\kappa)$. A method of confirmation is given by choosing $\lambda(\kappa)$. If $\lambda(\kappa) = \lambda = \text{const.}$, the method is called the method of the first kind and the confirmation function determined in this way may be denoted by $\mathfrak{c}_\lambda$. The method $\mathfrak{c}_\infty = \lim_{\lambda \to \infty} \mathfrak{c}_\lambda$ gives the case of independent trials, i.e., $\mathfrak{c}_\infty(h_i, e_i) = \mathfrak{c}_\infty(h_i, t)$ and $\mathfrak{c}_\infty(e_Q, t) = 1/\kappa^s$ (t = tautological evidence); the method $\mathfrak{c}_0$ gives the reciprocal case: $\mathfrak{c}_0(e_Q, t) = 1/\kappa$ if $s_i = s$ and $= 0$ if $s_i < s$. The method $\mathfrak{c}^*$ [Carnap, Logical foundations of probability, University of Chicago Press, 1950; Philos. Sci. **12**, 72–97 (1945); these Rev. **12**, 664; **7**, 46] is the method of the second kind, i.e., the case in which the $\lambda(\kappa)$ depend on κ. We have $\lambda(\kappa) = \kappa$ for $\mathfrak{c}^*$.

The second part is devoted to "Comparison of the success of inductive methods". The successfulness of the method $\mathfrak{c}_\lambda$ for a given state-description is defined by means of mean square error. It is shown that for each state-description there is an optimum method and, conversely, for each method $\mathfrak{c}_\lambda$ there are state-descriptions for which $\mathfrak{c}_\lambda$ is the optimum one. The appendix is devoted to Wald's decision functions.

In this work logical notions play in reality no role. The language $\mathfrak{L}_N^\tau$ may be fully described in terms of Boolean algebras or in terms of fields of sets. E.g., $\mathfrak{L}_N^\tau$ is the field of all subsets of the cartesian product $X \times X \times \cdots \times X$ (N times) where X is a set of power $\kappa = 2^\tau$. Each state-description is a point of this product. The conditions C1–C11 may also be expressed in terms of Boolean algebras or fields of sets. (C1 and C2 are to be dropped.) If we replace everywhere the function $\mathfrak{c}$ by $m(h) = \mathfrak{c}(h, t)$, we obtain a normed measure (probability) which is submitted to conditions C6–C11, and thus we can say that this paper is concerned with a special kind of normed measures on the finite cartesian products of finite sets. Among those measures there are, of course, ones which are of great importance for the theory of probability.

J. Łoś (Toruń).

Freudenthal, H. On the foundations of geometry to be sought in intuition and abstraction. Gregorianum **32**, 425–433 (1951). (Latin)

L'A. pense qu'il est très certainement nécessaire d'attirer l'attention des géomètres sur les thèses, tant mathématiques que philosophiques, de P. Hoenen, thèses longuement dévelopées par ce dernier dans de nombreux ouvrages et articles. Les méthodes et les points de vue de Hoenen cadrent davantage avec celles des géomètres anciens, comme Euclid, qu'avec celles des modernes. Néanmoins l'A. pense que les axiomaticiens peuvent y trouver des aspects et des problèmes parfois nouveaux, en tous cas fort difficiles—comme de construire la géométrie à partir de la notion de "être partie de" comme notion première, l'espace n'étant nullement considéré comme un ensemble de points—et aussi des notions utiles aux mathématiciens et même aux physiciens, comme celle de "principe d'individuation", notion que les deux auteurs n'entendent, semble-t-il, pas tout-à-fait de la même manière.

L'A. défend le point de vue axiomatique selon lequel la géométrie peut en droit se construire par la seule logique formelle, sans recours à l'intuition spatiale, à partir d'axioms suffisamment forts et nombreux, sans qu'il soit besoin de faire attention au sens intuitif des notions intervenant dans ces axioms. Hoenen, au contraire, pense que ce recours à l'intuition spatiale, qu'il appelle "intuition de l'étendu", est inévitable; et certains de ses positions le rapprochent des intuitionnists de l'école de Brouwer.

Cependent, de l'avis du référent, le système de Hoenen est un système cohérent et original, méritant d'être étudié pour lui-même, des fondements intuitifs et logiques des mathématiques, et qu'il est impossible de ranger dans aucune école de mathématiciens modernes. *A. Appert.*

Hoenen, P. On the noetics of geometry: a reply to the commentary of the distinguished H. Freudenthal. Gregorianum **32**, 434–452 (1951). (Latin)

Réponses de l'A. à certaines objections de H. Freudenthal [voir le rapport précédent], et surtout à certaines interprétations, qu'il estime inexactes, de sa propre pensée. L'A. en prend occasion pour développer certaines de ses propres positions sur "l'intuition intellectuelle de l'étendu", en ajoutant de nombreux renvois à ses propres ouvrages et articles. L'étude de ces ouvrages semble nécessaire au référent pour qu'on puisse se faire une idée exacte du système de Hoenen, système impossible à développer, de façon satisfaisante, en quelques lignes. *A. Appert* (Angers).

ALGEBRA

Harshbarger, Boyd, and Davis, Lyle L. Latinized rectangular lattices. Biometrics **8**, 73–84 (1952).

$K(K-1)$ varieties are arranged into K sets of K blocks of $(K-1)$ varieties each in such a way that each set contains a complete replication. Moreover, the blocks in each set are numbered in such a way that the union of all blocks with the same number forms is also a complete replication. Furthermore, the columns of the first set give the varieties which do not occur together in a block. The designs can be constructed from complete sets of orthogonal Latin squares. The author gives a complete account of the construction and analysis of these designs and illustrates it by several examples. *H. B. Mann* (Columbus, Ohio).

Mann, Henry B. Some theorems on difference sets. Canadian J. Math. **4**, 222–226 (1952).

A set $a_1, \cdots, a_k$ of residues modulo v is a difference set if $a_i - a_j \equiv d \pmod{v}$ has exactly λ solutions for every $d \not\equiv 0 \pmod{v}$. Trivially $k(k-1) = \lambda(v-1)$. This paper extends results of the reviewer and H. J. Ryser [Duke Math. J. **14**, 1079–1090 (1947), and Canadian J. Math. **4**, 495–502 (1951); these Rev. **9**, 370; **13**, 312]. A number t is a multiplier

if $\{ta_i\} = \{a_j + s\}$ for some s. It is shown that if a multiplier t has even order modulo a prime divisor q of v, then $n = k - \lambda$ is a square or q^2 times a square. Further strong restrictions of this type are found. For the case of cyclic planes (i.e., $\lambda = 1$) let t_1, t_2, t_3, t_4 be four multipliers. Then if $t_1 + t_2 = t_3$ and if $t_3 \neq t_4$, then $t_1 + t_4$ is not a multiplier. Also if $t_1 - t_2 = t_3 - t_4$, then $(t_1 - t_2)(t_1 - t_3) \equiv 0 \pmod{v}$. *Marshall Hall.*

√ *Perlis, Sam. **Theory of matrices.** Addison-Wesley Press, Inc., Cambridge, Mass., 1952. xiv+237 pp. $5.50.

This book is a modern, but nevertheless elementary, presentation of the theory of canonical matrices and is devoted in the main to the study of seven equivalence relations (called RST relations, to avoid confusion with the concept of equivalent matrices) on certain classes of matrices. These are: (i) equivalence over an arbitrary field $\mathfrak{F}$; (ii) equivalence over $\mathfrak{F}[x]$; (iii) similarity over $\mathfrak{F}$; (iv) congruence over the real field $\mathfrak{R}$; (v) Hermitian congruence over the complex field $\mathfrak{C}$; (vi) orthogonal similarity over $\mathfrak{R}$; (vii) unitary similarity over $\mathfrak{C}$.

The chapter headings are: (1) Introductory concepts; (2) Vector spaces; (3) Equivalence, rank and inverses; (4) Determinants; (5) Congruence and Hermitian congruence; (6) Polynomials over a field; (7) Matrices with polynomial elements; (8) Similarity; (9) Characteristic roots; (10) Linear transformations. The order of presentation of the various topics is thus unusual for an elementary textbook and the arguments are consequently sometimes more sophisticated than is common in such works, but the book is thorough, methodical, and extremely readable. Several appendices demonstrate some of the applications of the theory to maxima and minima, systems of linear differential equations, small vibrations, and the like. Somewhat surprisingly, however, the applications to coordinate geometry are scarcely mentioned. One pleasing feature is the manner in which topics which lie ahead are suggested and described before the theory itself is presented.

Although the existence of polynomials $p_1(x)$, $p_2(x)$ such that H.C.F. $(f_1(x), f_2(x)) = p_1(x)f_1(x) + p_2(x)f_2(x)$ is demonstrated, no method of determining $p_1(x)$, $p_2(x)$ is given. An unwary student might not observe that on p. 37 the row operations $e_2 \to e_2 - e_1$, $e_3 \to e_3 - 2e_1$ are permissable, but that $e_2 \to e_2 - e_1$, $e_1 \to e_1 - e_2$ are not so. *D. E. Rutherford.*

Menger, Karl. Une théorie axiomatique générale des déterminants. C. R. Acad. Sci. Paris **234**, 1941–1943 (1952).

Independent axioms symmetric as to rows and columns are given for the determinant of a matrix over a commutative ring with identity element. For the real numbers, a real-valued function will be the determinant if: (1) it is lower semi-additive in each row (that is,

$$f(v_1, \cdots, v_i' + v_i'', \cdots, v_n) \leqq f(v_1, \cdots, v_i', \cdots, v_n) + f(v_1, \cdots, v_i'', \cdots, v_n))$$

and upper semi-additive in each column; (2) it is lower semi-homogeneous in each row ($f(\cdots, cv_i, \cdots) \leqq cf(\cdots, v_i, \cdots)$ for $c \geqq 0$) and upper semi-homogeneous in each column; (3) it is non-negative if two rows are identical and non-positive if two columns are identical; and (4) it has the value one for the unit matrix. A theorem asserts that property (1) alone implies additivity and that the function for a general matrix is the sum of the $n!$ values of the function for the matrices obtained by replacing all but one element in each row and column by zero. *W. Givens* (Knoxville, Tenn.).

Hruška, Václav. Remarque sur la note de M. Jiří Seitz dans le no. 4, 1950, p. 137 des "Aktuárské Vědy." Czechoslovak Math. J. **1(76)**, 3–4 (1951) = Čehoslovack. Mat. Ž. **1(76)**, 3–4 (1951).

A simple demonstration is given of the fact discovered by Jiří Seitz [Aktuárské Vědy **8**, 137–144 (1949); these Rev. **12**, 471] that if $c_1x_1 + c_2x_2 + \cdots + c_nx_n$ takes $\sum_1^n a'_{ik}x_ix_k$ into $\sum_1^n a_{ik}x_ix_k$, then the determinant of the leading $(r-1)$-rowed minor of the former is $-c_1^{-2}$ multiplied by the determinant obtained by bordering (first row and column) the leading r-rowed minor of the latter by $(0, c_1, c_2, \cdots, c_r)$.

B. W. Jones (Boulder, Colo.).

Frank, Evelyn. On certain determinantal equations. Amer. Math. Monthly **59**, 300–309 (1952).

With the aid of previous results on continued fractions due to Grommer and Frank, the author is able to construct simple proofs of some known results concerning the location of the zeros of certain determinantal equations of the form $P(z) = |A_{ij}| = 0$ where $A_{ij} = 0$ for $j \neq i, i \pm 1$ and for $j < 0$, and where $A_{ii} = a_iz + b_i$, $A_{i,i\pm1} = \text{const.}$ and to extend some results to certain more general determinantal equations.

M. Marden (Milwaukee, Wis.).

Basilewitch, Vladimir. Über ein Verfahren der sukzessiven Approximation bei linearen algebraischen Gleichungssystemen. Bull. Soc. Math. Phys. Serbie **3**, no. 1–2, 47–55 (1951). (Serbo-Croatian. German summary)

To solve a linear system with a symmetric positive definite matrix of coefficients, the author proposes an under- or over-relaxation of the Jacobi (ordinary) iteration. The system is $\sum_{\nu=1}^k a_{\mu\nu}x_\nu - u^{(0)}{}_\mu = 0$ $(\mu = 1, \cdots, k)$. Let $x^{(0)}{}_\nu = 0$ (all ν). Suppose the components $x^{(n)}{}_\nu$ of the nth approximant have been determined $(n = 0, 1, \cdots)$. Let $u^{(n)}{}_\mu = u^{(0)}{}_\mu - \sum_{\nu=1}^k a_{\mu\nu}x^{(n)}{}_\nu$. In the author's process, $x^{(n+1)}{}_\nu = x^{(n)}{}_\nu + (a_{\nu\nu})^{-1}u^{(n)}{}_\nu C_n$. If all $C_n = 1$, one has the Jacobi process, not always convergent. For suitable choices of the C_n, $x^{(n)}{}_\nu$ will converge to the solution as $n \to \infty$. The author gives a formula for the C_n which minimizes the quadratic form $F_{n+1} = \frac{1}{2}\sum_{\mu=1}^k u^{(n+1)}{}_\mu x^{(n+1)}{}_\mu$.

G. E. Forsythe (Los Angeles, Calif.).

Osborne, Elmer E. On matrices having the same characteristic equation. Pacific J. Math. **2**, 227–230 (1952).

Over an infinite perfect field F, two square matrices A and B will have the same characteristic equation if and only if there exists a nonsingular P with elements in F such that every polynomial in A and $N = A - P^{-1}BP$, each term of which contains N at least once, is nilpotent. The existence of P is proved by use of suitable rational canonical forms and sufficiency is established by showing that the characteristic equation of $A - \theta N$ does not depend on θ.

W. Givens (Knoxville, Tenn.).

Bellman, Richard. A note on scalar functions of matrices. Amer. Math. Monthly **59**, 391 (1952).

If $\phi(A)$ is a polynomial in the elements of the matrix A such that $\phi(AB) = \phi(BA)$ for all pairs A and B, then $\phi(A)$ is a polynomial in the coefficients of the characteristic equation of A. The proof involves a continuity argument from the case of distinct roots, when a diagonal form is used, and so appears to implicitly assume the complex field.

W. Givens (Knoxville, Tenn.).

Charrueau, André. Formules matricielles relatives aux complexes linéaires et aux faisceaux de complexes linéaires. C. R. Acad. Sci. Paris **234**, 2252–2254 (1952).

The formulas all refer to skew-symmetric matrices of order four. Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be the duals of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ $(\mathfrak{B}_i = \omega_i\mathfrak{A}_i^{-1}$

with $\omega_i^2=\det \mathfrak{A}_i$). Then $\mathfrak{A}_1\mathfrak{B}_2\mathfrak{A}_1=\omega_{12}\mathfrak{A}_1-\omega_1\mathfrak{A}_2$, where ω_{12} is the bilinear invariant of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ and this formula is given a geometric interpretation. *W. Givens.*

Freudenthal, Hans. Produkte symmetrischer und antisymmetrischer Matrizen. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 193–198 (1952).

Necessary and sufficient conditions that $A=HK$ with H and K either symmetric or skew symmetric are found in terms of the elementary divisors of A. Results are: 1) the factorization property is invariant under similarity transformation of A (A and $S^{-1}AS$ are called equivalent by the author, contrary to established usage); 2) for both factors symmetric, A may be arbitrary; 3) both symmetric, one positive definite, the real A must be similar to a real diagonal matrix; 4) positive definite symmetric and skew, A is (complex) similar to a pure imaginary matrix; 5) H symmetric and K skew, the elementary divisors of A occur in pairs of the same degree with roots of opposite sign except for those with zero root; and 6) both factors skew, the elementary divisors occur in pairs with the same root and degree the same or possibly differing by one if the root is zero.

The short proofs in modern notation depend on reductions to canonical forms. For H or K nonsingular, results are easily translated into classical theorems on pencils of bilinear forms [cf. P. Muth, Theorie und Anwendung der Elementartheiler, Teubner, Leipzig, 1899, §§9 and 10, or Hodge and Pedoe, Methods of algebraic geometry, v. I, Cambridge, 1947, Ch. IX; these Rev. **10**, 396].

W. Givens (Knoxville, Tenn.).

Nosarzewska, M. On a Banach's problem of infinite matrices. Colloquium Math. **2** (1951), 194–197 (1952).

An infinite matrix $b_{m,n}$ is a permutation of a matrix a_{ij} if there exists a one-to-one transformation

$$(m, n)=(m(i, j), n(i, j))$$

of the set of pairs of positive integers into itself such that $b_{m,n}=a_{m(i,j),n(i,j)}$. If $m(i, j)=i$ when $i=1, 2, \cdots$ so that each element remains in its original row, then b_{mn} is a row permutation of $a_{i,j}$. Column permutations are analogously defined. S. Banach asked whether each permutation $b_{m,n}$ of a_{ij} is representable as the product of a finite set of row and column permutations. It is shown that the answer is affirmative. Each permutation is the product of five permutations in which row and column permutations alternate.

R. P. Agnew (Ithaca, N. Y.).

Vermes, P. Non-associative rings of infinite matrices. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 245–252 (1952).

This seems to be the first report in print of a discovery made by Mary Tropper (née Barnett) in 1950. With products of infinite matrices being defined in the usual way, a product being said to exist when the appropriate series are convergent, the infinite symmetric matrix

$$C=\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{bmatrix}$$

has an infinite set of two-sided reciprocals or inverses. For each complex number a, the infinite symmetric matrix

$$R_a=\begin{bmatrix} a & 1-a & -1 & a & 1-a & -1 & a & \cdots \\ 1-a & -1+a & 1 & -a & -1+a & 1 & -a & \cdots \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ a & -a & 0 & a & 1-a & -1 & a & \cdots \\ 1-a & -1+a & 0 & 1-a & -1+a & 1 & -a & \cdots \\ -1 & 1 & 0 & -1 & 1 & 0 & 0 & \cdots \\ a & -a & 0 & a & -a & 0 & a & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{bmatrix}$$

is an inverse of C. Other examples are given and used as a basis for the construction of a family of infinite matrices which constitutes, under the usual definitions, a non-associative ring with a unit element. *R. P. Agnew.*

Leng, Sen-ming. The characteristic roots of a matrix. Duke Math. J. **19**, 139–154 (1952).

Known inequalities for the roots which depend explicitly on the elements of the given matrix are generalized and systematically studied to obtain necessary and sufficient conditions for the equality to hold. Among the results proved are: 1) If $\lambda=x+iy$ is in the field of values $\{uA\bar{u}'; u\bar{u}'=1\}$ of A, then

$$|y|\leqq[\tfrac{1}{2}n(n-1)]^{1/2}\max_{1\leqq i<j\leqq n}\tfrac{1}{2}|a_{ij}-\bar{a}_{ji}|,$$
$$|x|\leqq[\tfrac{1}{2}n(n-1)]^{1/2}\max_{1\leqq i<j\leqq n}\tfrac{1}{2}|a_{ij}+\bar{a}_{ji}|.$$

The first inequality holds when $A+A'$ is real and the second when $A+A'$ is pure imaginary with the equality then holding in either case if and only if the quantity after the maximum sign has the same value for all $i<j$ and, if $n>3$, the common value is zero. 2) If $\lambda=uA\bar{v}'$ for $u\bar{u}'=v\bar{v}'=1$, $r_{ij}>0$ and $r_{ij}'=(r_{ij})^{-1}$, then $|\lambda|\leqq\max R_i^{1/2}\cdot\max S_i^{1/2}$ where $R_i=\sum_{j=1}^n|r_{ij}a_{ij}|$ and $S_i=\sum_{j=1}^n|r'_{ji}a_{ji}|$. Equality is attained if and only if $R_1=\cdots=R_n$, $S_1=\cdots=S_n$, and, whenever $a_{ij}a_{ik}a_{il}a_{ij}\neq 0$, $\arg(a_{ij}a_{il})=\arg(a_{ik}a_{kj})$ and $r_{ij}r_{jl}=r_{ik}r_{kj}$. 3) For λ as in 2), $|\lambda|\leqq(\sum_{i,j=1}^n|a_{ij}|^2)^{1/2}$ with equality possible if and only if A is of rank zero or one. 4) Equality in 3), with λ in the field of values, holds if and only if $\arg(a_{ij}a_{jl})$ is independent of j and $|a_{ij}a_{jl}|=|a_{ik}a_{jl}|$.

A matrix is connected by rows if, after suitable independent permutations of rows and columns,

$$a_{1,1}a_{2,1}\cdots a_{h_1,1}a_{h_1,2}a_{h_1+1,2}\cdots a_{h_2,2}a_{h_2,3}\cdots a_{n,k}\neq 0,$$

connected by columns if A' is by rows and connected if by rows or by columns. 5) For λ in the field of values, r_{ij} as in 2), $p>1$ and $p^{-1}+(p')^{-1}=1$,

$$|\lambda|\leqq\max_i\left(\sum_j|r_{ij}a_{ij}|^{1/p}\right)^{1/p}\cdot\max_i\left(\sum_j|r'_{ji}a_{ji}|^{1/p'}\right)^{1/p'}.$$

If A is connected by rows and columns, equality holds if and only if: each of the sums is independent of i; α) $\arg(a_{ij}a_{jl})$ is independent of j, and β) $\arg(a_{ij}a_{kj})=\arg(a_{il}a_{kj})$ when they are defined; and γ) $r_{ij}r_{jl}$ is independent of j, and δ) $r_{ij}r_{kl}=r_{il}r_{kj}$ when the corresponding a_{ij} are non-zero. 6) With the hypotheses of 5),

$$|\lambda|\leqq\max_i\tfrac{1}{2}(|r_{i1}a_{i1}|+\cdots+|r_{in}a_{in}|+|r'_{1i}a_{1i}|+\cdots+|r'_{ni}a_{ni}|);$$

for A connected by rows and columns; equality if and only if α), β), γ), δ) hold and $r_{i_1i_2}r_{i_2i_3}\cdots r_{i_ki_1}=r_{i_1i_{k+1}}$ for some indices such that $a_{i_1i_2}\cdots a_{i_ki_{k+1}}\neq 0$.

7) If equality holds in 3) for a λ in the field of values, $|\lambda|=\sum_i|a_{ii}|$ equals the modulus of the only non-zero root.

8) For λ a root, the inequality in 5) can be improved by replacing the product of the maxima by the maximum of the (corresponding) products. For A connected by rows or by rows and columns, conditions for equality are then obtained. 9) If λ is a root, it lies in the circle

$$C_i: \quad |\lambda-a_{ii}| \leqq (\sum_{j\neq i} |r_i a_{ij}|^p)^{1/p} \cdot (\sum_{j\neq i} |r'_j a_{ji}|^{p'})^{1/p'}$$

for at least one i. 10) If r of the n circles C_i have no interior point in common with the rest, they contain at least r roots; if no boundary point is common, exactly r. If A has no column or row with all non-diagonal elements zero and if the C_i are not all tangent at one point, any $r<n$ of the circles which have no interior point in common with the rest contain in the union of their open interiors just r of the roots. 11) The inequalities of 5), 8), and 9) admit $p\to 1$, $p'\to\infty$ provided γ) is replaced by $r_{ij}/r_{kj}=|a_{ij}|/|a_{kj}|$ when $a_{ij}a_{kj}\neq 0$; 10) may also be suitably modified.

It appears to be only a misprint that exponents have been omitted on p. 144 in the condition for equality in Hölder's inequality. *W. Givens* (Knoxville, Tenn.).

Stein, P. A note on the bounds of the real parts of the characteristic roots of a matrix. J. Research Nat. Bur. Standards **48**, 106–108 (1952).

With M the maximum and m the minimum of the real parts of the characteristic roots of a square matrix $A=(a_{ij})$ and $u_1, u_2, \cdots, u_n$ a set of positive numbers, it is proved that

$$\text{(i)} \qquad M \leqq \max_r \left\{ \tfrac{1}{2}(a_{rr}+\bar{a}_{rr}) + \frac{1}{2u_r} \sum_{s=1,\, s\neq r}^{n} |a_{rs}+\bar{a}_{sr}|u_s \right\},$$

and

$$\text{(ii)} \qquad m \geqq \min_r \left\{ \tfrac{1}{2}(a_{rr}+\bar{a}_{rr}) - \frac{1}{2u_r} \sum_{s=1,\, s\neq r}^{n} |a_{rs}+\bar{a}_{sr}|u_s \right\}.$$

The proof depends on writing $A=H+iK$ with H and K Hermitian and using the fact that a root of H must lie in a circle determined by the elements of $U^{-1}HU$, with $U=\text{diag}\,(u_1, \cdots, u_n)$. Applications to special real matrices are given. *W. Givens* (Knoxville, Tenn.).

Cherubino, Salvatore. Sulle matrici riemanniane e semiriemanniane. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **10**, 429–447 (1951).

Let ω be a q by r complex matrix, where $2q\geqq r\geqq q$. Assume that (ω) has rank r and that there exists a skew matrix M with integral elements such that $\omega M\omega'=0$, $\gamma=i\omega M\bar{\omega}'$ is positive semidefinite of rank p. Then $r\geqq 2p$ and ω is a Riemann matrix if $r=2p=2q$. The author calls ω a semiriemannian matrix if $2q>q+p\geqq r$. The matrix M may be normalized by the usual unimodular transformations and corresponding canonical forms for ω obtained. Only this and other elementary properties of such matrices are derived in the paper. *A. A. Albert* (Chicago, Ill.).

Sce, Michele. Su una generalizzazione delle matrici di Riemann. II. Ann. Scuola Norm. Super. Pisa (3) **5**, 301–327 (1951).

[For part I see same Ann. **5**, 81–103 (1951); these Rev. **13**, 524.] Let $\mathfrak{R}$, $\mathfrak{R}'$, and $\mathfrak{F}$ be fields of complex numbers and let the conjugate operation induce an involution of $\mathfrak{F}$. An n-rowed square matrix W with elements in $\mathfrak{R}'$ is called a Weyl matrix over $\mathfrak{F}$ if there exists a matrix C with elements in $\mathfrak{F}$ such that $\bar{C}'=eC$, $e=\pm 1$, $G=WC$ is positive definite Hermitian. Let C_1 be an m-rowed diagonal matrix with elements in $\mathfrak{R}$, $\bar{C}_1=eC_1$, and R be an m by n matrix with elements in $\mathfrak{R}$ such that $RW=C_1R$. Then R is called a right generalized Riemann matrix. An ordinary Riemann matrix ω, the corresponding matrix (ω), the Omega matrices of the reviewer, and the period matrices of the multiple integrals of the first kind of an algebraic variety are all right generalized Riemann matrices. The author uses the structure theory of Weyl matrices to derive a structure theory for right generalized Riemann matrices. *A. A. Albert.*

Mahajani, G. S., Thiruvenkatachar, V. R., and Thawani, V. D. An application of Tschebyscheff polynomials to a problem in symmetric functions. Proc. Indian Acad. Sci., Sect. A. **35**, 211–223 (1952).

The authors state their problem as follows: Given that $x_1, \cdots, x_n$ are non-negative, non-decreasing, that their first $n-1$ elementary symmetric functions have fixed (but unspecified) values, and that maximum and minimum values of successive x's occur alternately; to find the limits of variability of the x_i. This seems to mean: To determine $a_1, \cdots, a_n$ which are non-negative, increasing, and such that the first $n-1$ elementary symmetric functions of $0, a_2, a_2, a_4, \cdots$ are the same as those of $a_1, a_1, a_3, a_3, \cdots$ (both sequences to n terms). It turns out that these conditions determine the ratios of the a_i uniquely and that the $(a_i/a_n)^2$, $i=1, \cdots, n-1$, are the roots between 0 and 1 of the equation $\sin(2n\cos^{-1}z)=0$. *A. Erdélyi.*

Permutti, Rodolfo. Determinazione di equazioni algebriche a gruppo di Galois supersolubile. Giorn. Mat. Battaglini (4) **4(80)**, 159–185 (1951).

In this paper, $p_1, p_2, \cdots, p_n$ are primes, K is a field, $a_1, a_2, \cdots, a_n$ are elements of K, G is a smallest normal extension of K which contains the p_i roots of unity and which contains elements $u_1, \cdots, u_n$ such that $u_1^{p_1}=a_1$, $u_2^{p_2}=a_2+u_1, \cdots, u_n^{p_n}=a_n+u_{n-1}$. It is proved that a necessary and sufficient condition for the Galois group of G relative to K to be supersolvable is that either $p_1=p_2=\cdots=p_n$ or $p_2=p_3=\cdots=p_n$ and p_1 divides p_2-1. *H. Levi.*

Spampinato, Nicolò. Le algebre complesse, commutative, dotate di modulo. Ricerche Mat. **1**, 3–19 (1952).

Commutative algebras with unit element over the complex numbers, of order $\leqq 5$, are tabulated, and remarks are made on algebras of higher order. *I. Kaplansky.*

Spampinato, N. Le funzioni totalmente derivabili nelle algebre complesse, dotate di modulo, commutative del quarto ordine. Ricerche Mat. **1**, 145–166 (1952).

The theory of analytic functions over an algebra is worked out in detail for the commutative algebras of order 4. *I. Kaplansky* (Chicago, Ill.).

Papy, Georges. Sur l'irréductibilité de certains sous-espaces des algèbres extérieures. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 295–310 (1952).

Let V be a vector space of dimension m over a field F. The homogeneous part $\overset{p}{\bigwedge} V$ of degree p of the exterior algebra $\bigwedge V$ is irreducible with respect to the general linear group. Let m be even and let $H\in\overset{2}{\bigwedge} V$ be of maximal rank; an automorphism of V which leaves H invariant is called symplectic. The intersection of certain subalgebras of $\bigwedge V$ is shown to be the maximal subalgebra elementwise invariant under the symplectic group, and to consist of all linear combinations over F of reduced powers of H. If the charac-

teristic of F is 0, the homogeneous part of degree p of $\bigwedge V/(H)$ is irreducible with respect to the symplectic group.

E. R. Kolchin (New York, N. Y.).

Abstract Algebra

Klein-Barmen, Fritz. Schwach distributive Pseudoverbände. Math. Ann. **124**, 309–315 (1952).

Various abstract generalizations of the lattice ("Verband") concept are discussed, following D. O. Ellis [Publ. Math. Debrecen **1**, 205–208 (1950); these Rev. **12**, 472]. A "pseudo-lattice" has by definition two binary, commutative, and associative operations. Two "weak distributive laws" on the n-ary operation $\prod^{\smile} x = x \cup \cdots \cup x$ are defined, which are necessary and sufficient for a finite pseudo-lattice to be a lattice; the sufficiency breaks down in the infinite case.

G. Birkhoff (Cambridge, Mass.).

Sholander, Marlow. Trees, lattices, order, and betweenness. Proc. Amer. Math. Soc. **3**, 369–381 (1952).

On considère un ensemble S muni d'une loi qui à deux éléments quelconques $a, b \in S$ associe un sous-ensemble $(a, b) \subset S$. Quatre axiomes, notés (S), (T), (U_1), (U_2) énonçant des propriétés de cette loi sont alors étudiés. Une loi satisfaisant à (S), (T), (U_1) définit un système qu'on peut appeler un "arbre" car il généralise à la fois la notion d'arbre en topologie analytique, la notion d'arbre de la théorie des graphes, et celle donnée par G. Birkhoff [Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948, p. 47; ces Rev. **10**, 673]. Une loi satisfaisant à (S), (T), (U_2) caractérise complètement les treillis distributifs ayant un plus petit et un plus grand élément. Pour un "arbre" on peut définir une opération ternaire qui à a, b, c associe l'élément (unique) d tel que $\{d\} = (a, b) \cap (b, c) \cap (c, a)$ et une relation ternaire qui entre les éléments a, b, c s'énonce: "$b \in (a, c)$". Dans la second partie du mémoire, l'auteur donne des systèmes d'axiomes permettant de caractériser entièrement cette opération ternaire et cette relation ternaire dans le cas des arbres et dans le cas des chaînes. Il donne finalement un système d'axiomes pour une opération ternaire, définissant ainsi une structure qui généralise à la fois celle de treillis et celle d'"arbre".

J. Riguet (Paris).

Matusima, Yataro. On some problems of Birkhoff. Proc. Japan Acad. **28**, 19–24 (1952).

In connection with problem 7 of G. Birkhoff [Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948; these Rev. **10**, 673], the author examines various sets of postulates weaker than the usual ones for a lattice. For instance, given a lattice L, to each $a \in L$ let correspond a non-void abstract set C_a, containing a distinguished element $c(a)$, such that C_a and C_b are disjoint if $a \neq b$, and define operations among the elements of the C_a by $x \cap y = c(a \cap b)$, and dually, for $x \in C_a$, $y \in C_b$. Then the set of all x in the set union of the C_a, $a \in L$, satisfies ($\overline{\text{L1}}$) $x \cap x = x \cup x$, (L2) commutative laws, (L3) associative laws, and

(L*4) $x \cap (x \cup y) = x \cup (x \cap y) = x^*$ independent of y,

(M) $x \cup (x \cup y) = x \cup y$, $x \cap (x \cap y) = x \cap y$.

Conversely, any system satisfying these five conditions can be so constructed. [But by example IV, a system satisfying them need not be a lattice.] The author's other result, on independence of distributivity, was already found by R. Croisot [Canadian J. Math. **3**, 24–27 (1951); these Rev. **12**, 472]; see also G. Szász [Math. Ann. **124**, 291–293 (1952); these Rev. **13**, 815].

P. M. Whitman.

Goldie, A. W. On direct decompositions. I. Proc. Cambridge Philos. Soc. **48**, 1–22 (1952).

Soit A une algèbre au sens de G. Birkhoff [Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., 1948, p. vii; ces Rev. **10**, 673]. Soit $\mathfrak{L}(A)$ le treillis des relations d'équivalences compatibles avec la structure de A (son plus petit élément est l'identité Δ et son plus grand élément l'équivalence universelle $A \times A$). R étant une relation binaire quelconque on notera $\bar{R}$ l'équivalence compatible engendrée par R, c'est-à-dire, l'intersection de ceux des éléments de $\mathfrak{L}$ qui contiennent R. On sait que l'extension des théorèmes classiques de raffinement pour les décompositions directes de groupes s'est effectuée dans deux directions. 1) Pour les algèbres A, que nous appellerons ici du type 1, telles que deux éléments de $\mathfrak{L}(A)$ sont toujours permutables. C'est la direction O. Ore, G. Birkhoff [cf. op. cit., chap. VI]. 2) Pour les algèbres A, que nous appellerons ici du type 2, telles qu'il existe une opération binaire $+$ telle que, quel que soit x, $0+x=x+0=x$. C'est la direction B. Jónsson, A. Tarski [cf. Direct decompositions of finite algebraic systems, Univ. Notre Dame, 1947; ces Rev. **8**, 560].

On peut se proposer de rechercher les propriétés caractéristiques des algèbres (de type plus général que 1 et 2) qui sont telles que les théorèmes de raffinement sont vrais pour elles. Ce mémoire peut être considéré comme un premier pas vers la solution de ce problème. L'auteur montre d'abord que l'on ne peut étendre le type 2 en remplaçant l'opération binaire $+$ par une opération n-aire après avoir introduit les définitions ci-dessous qui généralisent des notions connues. $(\rho_1, \cdots, \rho_n)$ est un ensemble complet de projecteurs de A si $\rho_1, \cdots, \rho_n$ sont des endomorphismes de A tels que $\rho_i\rho_j = 0$ si $i \neq j$, $\rho_i^2 = \rho_i$ et si, lorsque $b_1, \cdots, b_n \in A$ sont tels que $\rho_i(b_i) = b_i$, il existe un $x \in A$ unique tel que $\rho_i(x) = b_i$. $(R_1, \cdots, R_n) \in \mathfrak{L}^n$ est un ensemble complet d'équivalences facteurs de A si quels que soient i, j on a $R_iR_j = R_jR_i$,

$$\Delta = \overline{\bigcup_{k \neq i} R_k} \cap R_i, \quad A \times A = \overline{\bigcup_{1 \leq k \leq n} R_k}.$$

$(B_1, \cdots, B_n)$ est un ensemble complet de sous-algèbres facteurs de A s'il existe un ensemble complet $(\rho_1, \cdots, \rho_n)$ de projecteurs de A tels que $B_i = \rho_i(A)$. (Cette définition convient car on montre qu'elle équivaut à l'isomorphie de A avec le produit direct de $B_1, \cdots, B_n$.)

Soient Π (respectivement Π', respectivement Π'') l'ensemble des ensembles complets de projecteurs de A (respectivement l'ensemble des ensembles complets d'équivalences facteurs de A, respectivement l'ensemble des ensembles complets de sous algèbres facteurs de A). Alors la correspondance qui transforme $(\rho_1, \cdots, \rho_n) \in \Pi$ en $(R_1, \cdots, R_n) \in \Pi'$ où $R_i = \bigcap_{j \neq i} \rho_j^{-1} \rho_j$ est une application biunivoque de Π sur Π' dite canonique dont l'inverse transforme $(R_1, \cdots, R_n) \in \Pi'$ en $(\rho_1, \cdots, \rho_n) \in \Pi$ où ρ_i est tel que $\{\rho_i(x)\} = \overline{\bigcup_{j \neq i} R_j(x)} \cap R_i(0)$. Ceci précise le théorème 4 de G. Birkhoff [op. cit. p. 87]. Mais la correspondance qui transforme $(\rho_1, \cdots, \rho_n) \in \Pi$ en $(B_1, \cdots, B_n) \in \Pi''$ où $B_i = \rho_i(A)$ est une application de Π sur Π'' non biunivoque en général. Elle le devient cependant (et on en deduit alors une triple correspondance canonique entre Π, Π', Π'') lorsque $B_i = \rho_i(A)$ où ρ_i est un projecteur régulier, c'est-à-dire, tel que les deux équivalences $\rho_i^{-1}\rho_i$ et $\overline{B_i \times B_i}$ permutent. Quand A est du type 1 ou du type 2,

tout projecteur est régulier. On définit sur Π une relation d'ordre appelée raffinement, une autre appelée contraction, et une relation d'équivalence appelée échange. Ces relations se transportent sur Π' et sur Π'' par la triple correspondance canonique.

La seconde partie du mémoire généralise les méthodes et les concepts introduits par H. Fitting pour l'étude des décompositions directes des groupes: l'introduction de la notion d'endomorphisme normal permet de démontrer un énoncé analogue au lemme de Fitting, alors que la notion d'addition des endomorphismes doit être remplacée par celle d'"ensemble uni d'endomorphismes". *J. Riguet.*

Goldie, A. W. On direct decompositions. II. Proc. Cambridge Philos. Soc. **48**, 23–34 (1952).

On applique ici la théorie développée dans le mémoire analysé ci-dessus aux algèbres de type 1. On définit d'abord un certain élément de $\mathfrak{L}(A)$ appelé équivalence centrale directe de A et dont la définition, trop longue pour être donnée ici, s'inspire directement de Kurosh [Izvestiya Akad. Nauk SSSR. Ser. Mat. **10**, 47–72 (1946); ces Rev. **8**, 309]. En adaptant alors les méthodes de Kurosh [op. cit.], de R. Baer [Bull. Amer. Math. Soc. **54**, 167–174 (1948); ces Rev. **9**, 410], et de B. Jónsson et A. Tarski [op. cit.] l'auteur démontre que si le sous-treillis de $\mathfrak{L}(A)$ constitué par les éléments de $\mathfrak{L}(A)$ inclus dans l'équivalence centrale directe satisfait aux deux conditions des chaînes, le théorème de raffinement est vrai pour A. *J. Riguet* (Paris).

Devidé, Vladimir. Über Transformationen isomorpher algebraischen Operationen. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **7**, 3–6 (1952). (Serbo-Croatian. German summary)

Soit S un ensemble muni d'une loi ϕ partout définie. Le nombre de lois sur S isomorphes à ϕ est égal à l'index du groupe d'automorphismes de ϕ dans le groupe symétrique de S. [Note du rapporteur. Ceci est évident et est vrai pour toute structure.] *J. Riguet* (Paris).

Vagner, V. V. On the theory of partial transformations. Doklady Akad. Nauk SSSR (N.S.) **84**, 653–656 (1952). (Russian)

$\mathfrak{M}(A\times A)$ étant l'ensemble des relations biunivoques de A dans A (c'est-à-dire, des relations $R\subset A\times A$ telles que $RR^{-1}\cup R^{-1}R\subset\Delta$), la relation d'inclusion de deux relations de $\mathfrak{M}$ et la propriété pour une relation de $\mathfrak{M}$ d'être symétrique peuvent se caractériser en faisant seulement intervenir la composition des relations. Dans la seconde partie de cette note, l'auteur démontre que si B est un sous ensemble de A ayant même puissance que son complémentaire, toute relation biunivoque de B dans B peut être prolongée en une permutation de A (plus précisement: si $\mathfrak{S}(A\times A)$ est le groupe symétrique de A, $\mathfrak{M}(B\times B)=\Delta_B\mathfrak{S}(A\times A)\Delta_B$). *J. Riguet* (Paris).

Vorob'ev, N. N. On ideals of associative systems. Doklady Akad. Nauk SSSR (N.S.) **83**, 641–644 (1952). (Russian)

On entend ici par système associatif un ensemble muni de la structure déterminée par une loi partout définie associative. L'auteur énonce sans démonstrations neuf théorèmes relatifs aux L systèmes associatifs (c'est-à-dire, aux systèmes associatifs pour lesquels le treillis des idéaux à gauche satisfait à la condition minimale) en utilisant les concepts de composante idéale à gauche (=ensemble des générateurs d'un idéal à gauche principal) et de facteur correspondant à une composante idéale [ce dernier concept est emprunté à D. Rees, Proc. Cambridge Philos. Soc. **36**, 387–400 (1940); ces Rev. **2**, 127]. Le premier théorème généralise un résultat de J. A. Green [Ann. of Math. **54**, 163–172 (1951); ces Rev. **13**, 100], les trois derniers théorèmes précisent en certains points les recherches de E. S. Liapin [Mat. Sbornik **20(62)**, 497–515 (1947); ces Rev. **9**, 134; l'auteur cite aussi un oeuvre inaccessible dans les Učenye Zapiski Leningrad. Gos. Pedagog. Inst. im. Gercen. **89** (1951)]. *J. Riguet.*

Smiley, M. F. Notes on left division systems with left unit. Amer. J. Math. **74**, 679–682 (1952).

A left division system is defined to be an algebra with one binary operation such that, for each a and b of the algebra, $xa=b$ is uniquely solvable. A normal subsystem is defined to be the kernel of a homomorphism. The author proves Zassenhaus' lemma for left division systems with left units in the following form. If A, B, A_1, and B_1 are subsystems of a system G, and if A_1 is a normal subsystem of A, and B_1 of B; then $A_1(A\cap B_1)$ is a normal subsystem of $A_1(A\cap B)$, $B_1(B\cap A_1)$ is a normal subsystem of $B_1(B\cap A)$, and the identity mapping of G induces an isomorphism of the corresponding quotient systems. *H. A. Thurston* (Bristol).

Teissier, Marianne. Sur l'algèbre d'un demi-groupe fini simple. C. R. Acad. Sci. Paris **234**, 2413–2414 (1952).

The author considers the algebra A over a field $\mathfrak{F}$ of characteristic 0 obtained by taking as base elements the elements of a finite simple semi-group S without zero and multiplication table the same as that of S. She shows that if G_1 is a minimal left ideal of S (and hence the sum of disjoint groups isomorphic to a group C), the subalgebra A_1 of A with the elements of G_1 as base elements has radical $R_1\neq 0$, providing that G_1 is not a group. Further, $A_1R_1=0$ and A_1-R_1 is isomorphic to the group algebra of C.

D. Rees (Cambridge, England).

Teissier, Marianne. Sur l'algèbre d'un demi groupe fini simple. II. Cas général. C. R. Acad. Sci. Paris **234**, 2511–2513 (1952).

This note continues the paper reviewed above. It is shown that if S is any simple semi-group without zero which is not a group, the algebra A of S over a field of characteristic 0 has a non-zero radical R satisfying $ARA=0$. *D. Rees.*

Rédei, L., und Steinfeld, O. Über Ringe mit gemeinsamer multiplikativer Halbgruppe. Comment. Math. Helv. **26**, 146–151 (1952).

For any ring T denote by T^+ and $T^\times$, respectively, T considered as a group under addition and as a semigroup under multiplication. Let R be the ring of the integers mod p^e where p is an odd prime. If $e\neq 2$ and S is a ring with $S^\times$ isomorphic to $R^\times$, then S is isomorphic to R. If $e=2$, there exists a ring S not isomorphic to R but with $S^\times$ isomorphic to $R^\times$; any third ring T with $T^\times$ isomorphic to $R^\times$ is isomorphic either to R or to S. The authors raise the general question of the extent to which T is determined by $T^\times$ and, in addition to the theorem stated above, consider cases arising in algebraic number fields. *R. M. Thrall.*

Herz, Jean-Claude. Sur les idéaux semi-premiers ou parfaits. Étude des propriétés latticielles des idéaux semi-premiers. C. R. Acad. Sci. Paris **234**, 1515–1517 (1952).

An ideal I in a commutative ring A is called semi-prime if it coincides with its radical [Krull, Math. Ann. **101**, 729–744 (1929)]. The paper first proves a theorem of Fuchs

[Norske Vid. Selsk. Forh., Trondhjem **20**, 112–114 (1947); these Rev. **11**, 310] that I is semi-prime if, and only if, it contains the intersection of two ideals whenever it contains their product, without using the fact that a semi-prime ideal is an intersection of prime ideals and thus without using the axiom of choice. Semi-prime ideals form a distributive lattice. Further, $\{B \cup C\} \cap \{B \cup D\} = \{B \cup CD\}$ for any subsets B, C, D of A, where $\{X\}$ denotes the smallest semi-prime ideal containing a subset X of A.

T. Nakayama (Nagoya).

Braconnier, Jean. Sous-modules d'un module complètement décomposable. Ann. Univ. Lyon Sect. A. (3) **14**, 29–33 (1951).

The author proves the following theorem, due to Koulikoff [Mat. Sbornik **16(58)**, 129–162 (1945); these Rev. **8**, 252]. Let A be a principal ideal ring, and M a direct sum of cyclic A-modules; then any submodule of M is again a direct sum of cyclic modules. The proof is similar to one given by Dieudonné [Portugaliae Math. **11**, 1–5 (1952); these Rev. **13**, 720]. *I. Kaplansky* (Chicago, Ill.).

Schöneborn, Heinz. Über Linearformenmoduln unendlichen Ranges. I. Primäre, kompakte Linearformenmoduln. J. Reine Angew. Math. **189**, 168–185 (1951).

Let M be the Cartesian product of a countable number of copies of the p-adic integers. The author studies the structure of M, regarded as a topological module over the p-adic integers. Topics include bases, linear transformations, and the classification of closed submodules. In particular, a criterion is given to determine when a closed submodule admits a basis consisting of multiples of a basis of M. [Reviewer's remark. Additional light is shed on the subject by Pontrjagin's duality, for M is a compact group whose character group is the direct sum of a countable number of copies of the group of type p^∞. The criterion mentioned above turns out to be equivalent to Prüfer's theorem that a countable primary group with no elements of infinite height is a direct sum of cyclic groups.] *I. Kaplansky.*

Schöneborn, Heinz. Über Linearformenmoduln unendlichen Ranges. II. Nichtarchimedisch perfekt bewertete, operatorreduzierte Linearformenmoduln. J. Reine Angew. Math. **189**, 193–203 (1952).

In the first part of the paper the topic is, in the terminology of Lefschetz, linearly compact vector spaces. Among other things, questions of change of basis are examined. In the second part the author studies R, the Cartesian product of copies of the p-adic numbers. The module M of the preceding paper is a closed submodule, and R can be studied in terms of M and R/M. Finally, the closed submodules of R are classified. *I. Kaplansky* (Chicago, Ill.).

Albert, A. A. On simple alternative rings. Canadian J. Math. **4**, 129–135 (1952).

It is shown that a simple alternative ring which contains an idempotent not its unity quantity is either associative or is a Cayley algebra over its center. This result is comparable to the theorem on alternative division rings proved by Bruck and Kleinfeld and also by Skornyakov.

Given an idempotent u, not the unity quantity, in any alternative ring C, there will be a two-sided Peirce decomposition of C into the module direct sum $C_{11}+C_{10}+C_{01}+C_{00}$, where $uC_{ij}=iC_{ij}$, $C_{ij}u=jC_{ij}$, $i, j=0, 1$. A number of properties of these modules are established, such as $C_{11}C_{00}=C_{00}C_{11}=0$, which is trivial if C is associative. Also relations between ideals of C_{ii} and ideals of C are found. These results are applied to the case in which C is simple but not associative. First $C=C_{10}C_{01}+C_{10}+C_{01}+C_{01}C_{10}$. Then it is shown that C_{11} and C_{00} are isomorphic fields with units e_{11} and e_{00}. From these units further elements are constructed showing that C is an eight-dimensional Cayley algebra.

Marshall Hall (Columbus, Ohio).

Albert, A. A. New simple power-associative algebras. Summa Brasil. Math. **2**, 183–194 (1951).

This paper is concerned with the construction of noncommutative simple power-associative algebras. The construction is based on the following general construction. Suppose that $\mathfrak{G}$ is an arbitrary algebra, J an involution on $\mathfrak{G}$, and T_0 an arbitrary linear transformation on $\mathfrak{G}$. Further, suppose that $S(\mathfrak{G})$ is the linear space of J-symmetric elements of $\mathfrak{G}$. Then if x, y belong to $S(\mathfrak{G})$ and if $T=T_0+T_0J$, $xoy=\frac{1}{2}(xy+yx)+(xy-yx)T$ also belongs to $S(\mathfrak{G})$. Hence this construction defines an algebra $B(\mathfrak{G}, t)$ with vector space $S(\mathfrak{G})$. The algebra $B(\mathfrak{G}, T)$ is said to be bonded to the commutative algebra (also denoted by $S(\mathfrak{G})$) whose elements are those of the space $S(\mathfrak{G})$ with product $x \cdot y=\frac{1}{2}(xy+yx)$. It is shown that if $\mathfrak{G}$ is power-associative, so is $B(\mathfrak{G}, T)$ and powers in $B(\mathfrak{G}, T)$ and $S(\mathfrak{G})$ coincide.

The following result is also proved. Denote by $\mathfrak{F}$ a field of characteristic prime to 30. Let $\mathfrak{A}$ be an algebra over $\mathfrak{F}$ such that $x^2 \cdot x=x \cdot x^2$ in $\mathfrak{A}$. Suppose further that $\mathfrak{A}^{(+)}$ is a central simple special Jordan algebra $S(\mathfrak{G})$ of degree >2 (so that $\mathfrak{G}$ is associative). Then $\mathfrak{A}$ is also power-associative and central simple and for a suitable extension K of $\mathfrak{F}$, $\mathfrak{A}_K$ is bonded to $S(\mathfrak{G}_K)$. A similar result is proved in the case where $\mathfrak{A}^{(+)}$ is the exceptional simple Jordan algebra. Finally it is deduced that, if $\mathfrak{A}$ satisfies $x^2 \cdot x=x \cdot x^2$ and $(x^2 \cdot x)x=x^2 \cdot x^2$, and if also $\mathfrak{A}^{(+)}$ is central simple of degree $t>2$, then $\mathfrak{A}$ is power-associative (i.e., the usual restriction of the characteristic of $\mathfrak{F}$ to zero characteristic can be replaced by the restriction to characteristic prime to 30.) *D. Rees.*

Kokoris, L. A. Power-associative commutative algebras of degree two. Proc. Nat. Acad. Sci. U. S. A. **38**, 534–537 (1952).

A. A. Albert has shown that, over a field of characteristic $\neq 2, 3, 5$, every simple power-associative commutative algebra of degree greater than two is a classical Jordan algebra [Trans. Amer. Math. Soc. **69**, 503–527 (1950); these Rev. **12**, 475]. In his dissertation [Univ. Chicago, 1952] the author extended this result to include characteristics 3 and 5. In this note, for each prime $p>5$, examples are given of simple power-associative commutative algebras of degree two and characteristic p which are not Jordan algebras. The examples are of dimensions $3p$ and $4p$, and are not Jordan algebras because they are not stable. The algebra of dimension $4p$ is u-stable for a certain idempotent u.

R. D. Schafer.

Amitsur, Shimshon. The problem of Kurosh-Levitzki-Jacobson. Riveon Lematematika **5**, 41–48 (1952). (Hebrew. English summary)

The problem of Kurosh is whether an algebraic algebra A is locally finite (locally finite means that finitely generated subalgebras are finite-dimensional). When A satisfies a polynomial identity, the question was answered in the affirmative by Kaplansky, using topological methods [Trans. Amer. Math. Soc. **68**, 62–75 (1950); these Rev. **11**, 317]. An algebraic proof was given by Levitzki, using some struc-

ture theory from an unpublished paper. The author gives a third proof, this one direct and algebraic. An algebraic proof is also given for the following theorem, first proved topologically by Kaplansky [Trans. Amer. Math. Soc. **70**, 219–255 (1951); these Rev. **13**, 48]: If every algebraic division algebra over a field F is locally finite, then every algebraic algebra of bounded index (i.e., the indices of nilpotence of its elements are bounded) is locally finite. The paper also contains a history of the problem. *I. S. Cohen.*

Dixmier, J. Sur un théorème d'Harish-Chandra. Acta Sci. Math. Szeged **14**, 145–156 (1952).

The author gives a simplified proof of the following extended version of an important theorem due to Harish-Chandra [Trans. Amer. Math. Soc. **70**, 28–96 (1951), Th. 4; these Rev. **13**, 428]. Let L be a (complex) Lie algebra of the form $M \oplus N$, where M is an ideal and N is semi-simple. Let $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$ be the respective enveloping algebras, $\mathfrak{M}$ and $\mathfrak{N}$ being identified in the natural way with subalgebras of $\mathfrak{L}$. Let $\mathfrak{C}$ be the center of $\mathfrak{L}$, and $\mathfrak{D}$ the subalgebra of $\mathfrak{M}$ consisting of elements commuting with $\mathfrak{N}$. Let f be a finite-dimensional representation of N, let h be a homomorphism of $\mathfrak{D}$ into the complex field. The conclusion states: 1) there exist only a finite number of inequivalent irreducible representations ϕ of $\mathfrak{L}$ which contain f when restricted to N and such that $\phi(x) = h(x)1$ (where 1 = unit operator) for x in $\mathfrak{D}$; 2) for any such representation ϕ, the representation space is the direct sum of finite-dimensional subspaces each of which is invariant under the restriction of ϕ to N, different subspaces arising from inequivalent, and in fact disjoint, representations; 3) $\phi(x)$ is a scalar for x in $\mathfrak{C}$. As the author states, the proof, which is purely algebraic, is basically along the same lines as Harish-Chandra's. *I. E. Segal.*

Theory of Groups

Vagner, V. V. Generalized groups. Doklady Akad. Nauk SSSR (N.S.) **84**, 1119–1122 (1952). (Russian)

L'auteur appelle groupe généralisé un demi-groupe (c'est-à-dire, un ensemble S muni d'une loi partout définie associative) dont tous les idempotents sont deux à deux permutables et dont tout élément admet un inverse généralisé (pour tout $s \in S$ un élément $\bar{s} \in S$ tel que $s\bar{s}s = s$ et $\bar{s}s\bar{s} = \bar{s}$ est nécessairement unique et est appelé inverse généralisé de s). Si $\mathfrak{S}$ est un ensemble de relations biunivoques pour lequel, quels que soient $S_1 \in \mathfrak{S}$ et $S_2 \in \mathfrak{S}$, on ait (*) $S_1S_2 \in \mathfrak{S}$ et $S_1^{-1} \in \mathfrak{S}$, $\mathfrak{S}$ est un groupe généralisé, si on considère que $\mathfrak{S}$ est un demi-groupe pour la composition des relations. Tout groupe généralisé est isomorphe à un groupe généralisé constitué, d'après ce qui précède, par un ensemble de relations biunivoques satisfaisant à (*). Par addition de conditions supplémentaires à la définition de groupe généralisé on peut retrouver des généralisations connues de la notion de groupe, en particulier, la notion de groupoïde de Brandt. *J. Riguet* (Paris).

Dubreil, Paul. Contribution à la théorie des demi-groupes. II. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **10**, 183–200 (1951).

This paper is the continuation of an earlier paper by the same author [Mém. Acad. Sci. Inst. France **63**, no. 3 (1941); these Rev. **8**, 15]. The first chapter is concerned with various properties of principal equivalences on a demi-group D. These are associated with a complex H of D, the right principal equivalence R_H being defined by $x \equiv y\ (R_H)$ if $(H:x)_d = (H:y)_d$ ($(H:x)_d$ consists of all elements z of D such that $xz \in H$). The residue W_H of H consists of all elements x of D such that $H:x$ is empty. This is certainly a right ideal of D. The author considers under what conditions a given right ideal of D is the residue of some complex H. He further considers conditions on complexes H_1, H_2 which will imply $R_{H_1} \cap R_{H_2} = R_H$ where $H = H_1 \cap H_2$. Such a set of conditions is that H_1, H_2 are strong (i.e., $(H_i:x)_d \cap (H_i:y)_d \neq 0$ implies $(H_i:x)_d = (H_i:y)_d$) and $W_H = W_{H_1} = W_{H_2}$. In the case where H_1, H_2 are strong subgroupoids of D, he also considers conditions which will imply that R_{H_1}, R_{H_2} are permutable.

The second chapter is chiefly concerned with an equivalence relation Σ_H defined as follows. $x \sigma_H y$ means that there exist h, h' in H such that $hx = h'y$. Then Σ_H is the transitive closure of the relation σ_H. A number of results are obtained which are of too technical a nature to give in detail here. *D. Rees* (Cambridge, England).

Thierrin, Gabriel. Sur les demi-groupes inversés. C. R. Acad. Sci. Paris **234**, 1336–1338 (1952).

A demi-group D is said to be inversed on the right if, for each x in D, there exists in D an element x' such that xx' is idempotent. If, at the same time, D satisfies both cancellation laws, then D is a group. If, for some y in D, yy' is idempotent and in the center of D, then Dyy' is a demi-group inversed on the right, yy' is the identity element of Dyy', and D is homomorphic to Dyy'. *F. Kiokemeister.*

Schwarz, Štefan. On the structure of simple semigroups without zero. Czechoslovak Math. J. **1(76)**, 41–53 (1951) = Čehoslovack. Mat. Ž. **1(76)**, 51–65 (1951).

The author studies the effect of the following restrictions on a semi-group S without zero: (A) S contains at least one minimal left ideal; (B) S contains both minimal left ideals and minimal right ideals. He shows that if (A) holds, S is the sum of its minimal left ideals if and only if it is simple. Further, if S is simple and satisfies (B), every minimal left (right) ideal is of the form eS (Sf) where e, f are idempotents, and S is the sum of disjoint isomorphic groups. Also, if (A) holds in a simple semi-group S (without zero), (B) holds if and only if S contains an idempotent. Finally he shows that if S is any semi-group satisfying (A) and if one minimal left ideal of S contains an element of finite order, then (i) S contains minimal right ideals, (ii) every minimal left (right) ideal of S is of the form eS (Sf), where e, f are idempotent, (iii) the union of all minimal left ideals of S coincides with the union of all minimal right ideals, this union, K, being the Suschkewitsch kernel of S. *D. Rees.*

Weaver, Milo W. Cosets in a semi-group. Math. Mag. **25**, 125–136 (1952).

A semi-group S contains S' as a left divisor semi-group if S is the union of disjoint left cosets cS' and these cosets are independent of the c's. If, furthermore, $aS' = S'a$ for every a in S, then S' is called a divisor semi-group, and the quotient semi-group S/S' is defined by coset multiplication in the case that $S'S' = S'$. If S' and S/S' are groups, then S is a group. Conditions are given that a subsemi-group S' should be a left divisor semi-group and that S' should be at the same time a group. The subsemi-groups of the multiplicative semi-group of the ring of rational integers modulo an element m are discussed. The uniqueness of factorization in these semi-groups is established. *F. Kiokemeister.*

Permutti, Rodolfo. Determinazione dei gruppi finiti in omomorfismo di struttura con un gruppo quadrinomio. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **9**, 237–246 (1950).

A finite group G is lattice-homomorphic to a quadrinomial group if and only if G is the direct product of a group of odd order and either the quaternions or a quadrinomial group. [For more general results, see Zappa, Ricerche Mat. **1**, 78–106 (1952); these Rev. **13**, 906.] *P. M. Whitman.*

***Mal'cev, A. I. On the completion of group order.** Trudy Mat. Inst. Steklov., v. 38, pp. 173–175. Izdat. Akad. Nauk SSSR, Moscow, 1951. (Russian) 20 rubles.

Generalizing theorems of H. Šimbireva [Mat. Sbornik **2**(62), 145–178 (1947); these Rev. **8**, 563] and A. A. Vinogradov [Partial orderability and structural orderability of groups, Thesis, Moscow, 1949 (unavailable)], the author proves that in every locally nilpotent partially ordered group without elements of finite order the order can be expanded to the whole group. *H. Freudenthal.*

Rédei, L. Die Anwendung des schiefen Produktes in der Gruppentheorie. J. Reine Angew. Math. **188**, 201–227 (1950).

The author has defined a "skew product" in an earlier paper [Comment. Math. Helv. **20**, 225–264 (1947); these Rev. **9**, 131]. This appears now as a special case of a general construction which we shall treat here only in the binary case. Let G and Γ be two groups with elements $a, b, \cdots$ and $\alpha, \beta, \cdots$ and units e and ϵ. From the pairs (a, α) a new algebraic structure is formed, the composition being defined by $(a, \alpha)(b, \beta)=(f, \varphi)$ where $f \in G$ and $\varphi \in \Gamma$ are one-valued functions of the four arguments a, b, α, β. This is the author's "skew product" of G and Γ, but he deals explicitly only with the case where (in operator notation) $f=ab^{\alpha}\beta^{a}$, $\varphi=a^{b}\alpha^{b}\beta$ (it is of course assumed that $b^{\alpha}, \beta^{a} \in G$ and $a^{b}, \alpha^{b} \in \Gamma$) and writes then $G \circ \Gamma$. The first question that arises naturally is under what conditions $G \circ \Gamma$ is again a group. An elementary calculation concerned with the validity of the associative law, the permissible choice of (e, ϵ) as unit-element, etc. leads to a long system of 16 conditions which the author shows to be necessary and sufficient for $G \circ \Gamma$ to form a group. The skew product $G \circ \Gamma$ contains for suitable choices of f and φ a number of special cases some of which are well known. If, of the four relations $b^{\alpha}=b$, $\beta^{a}=e$, $a^{b}=\epsilon$, $\alpha^{b}=\alpha$, exactly k are assumed to hold, then the author says that $G \circ \Gamma$ has k-fold degeneracy. $k=4$ leads to the direct product of G and Γ. Apart from this the author lists 11 cases of which we mention here only the most interesting.

(1) $f=ab$, $\varphi=a^{b}\alpha^{b}\beta$, $k=2$. In this case the skew product turns out to be an extension of Γ by G. The group conditions specialise here to a form equivalent to Schreier's extension theory. The problem of isomorphism of extensions (for different choices of the analogue to Schreier's automorphism and factor-system) is solved theoretically, but remains in practice as inaccessible as it is in the classical theory.

(2) $f=ab^{\alpha}$, $\varphi=\alpha^{b}\beta$, $k=2$. In this case the elements of G are uniquely representable in the form $g\gamma$ with $g \in G$, $\gamma \in \Gamma$, i.e., G is "factorisable" by G and Γ. (Cf. the series of papers on factorisable groups reviewed below.) The group conditions lead here to Zappa's main result on factorisable groups.

(3) Let Γ be the additive group of a ring P and put $f=ab$, $\varphi=\alpha b'+\beta$, $b' \in P$. This type (which has $k=3$) comprises groups which are non-Abelian, but have only Abelian subgroups. (Whether infinite groups of this kind exist is an open problem.) These groups have been completely classified by the author [loc. cit.] with, as he puts it, insurpassable elegance. If they are p-groups, then all subgroups of index p must be Abelian, and the structure of such groups is comparatively simple. If they are not p-groups, then their orders are of the form $p^{u}p^{v}$ where p and q are distinct primes and v the smallest integer for which $p \mid q^{v}-1$. If one chooses for G a cyclic group of order p^{u}, generated by g, and for the ring P the Galois field $GF(q^{v})$ with an element $\rho \neq \epsilon$, $\rho^{p}=\epsilon$, then the group in question can be represented as skew product with the multiplication $(g^{i}, \alpha)\ (g^{k}, \beta)=(g^{i+k}, \alpha\rho^{k}+\beta)$. The author also gives an example of a "non-degenerate" case ($k=0$) of $G \circ \Gamma$. Let p be a prime, $R(m)$ the residue-class ring (mod m) and put both G and Γ equal to the additive group of $R(p^{3})$. Define multiplication by

$$(a, \alpha)(b, \beta)=(a+b+p\alpha(b+\beta),\ \alpha+\beta+pb(a+\alpha)),$$

the integers a, b, α, β being taken (mod p^{3}). This yields a non-degenerate group of order p^{4}. *K. A. Hirsch.*

Szép, J., and Rédei, L. On factorisable groups. Acta Univ. Szeged. Sect. Sci. Math. **13**, 235–238 (1950).

Szép, J. On factorisable, not simple groups. Acta Univ. Szeged. Sect. Sci. Math. **13**, 239–241 (1950).

Rédei, L. Zur Theorie der faktorisierbaren Gruppen. I. Acta Math. Acad. Sci. Hungar. **1**, 74–98 (1950). (German. Russian summary)

Szép, J. On factorisable simple groups. Acta Sci. Math. Szeged **14**, 22 (1951).

Szép, J. Zur Theorie der faktorisierbaren Gruppen. Publ. Math. Debrecen **2**, 43–45 (1951).

Itô, Noboru. Remarks on factorizable groups. Acta Sci. Math. Szeged **14**, 83–84 (1951).

Szép, J. Zur Theorie der endlichen einfachen Gruppen. Acta Sci. Math. Szeged **14**, 111–112 (1951).

This series of papers deals with groups G which can be written in the form $G=HK$, where H and K are proper subgroups of G. G is then said to be factorisable by H and K. The representation of the elements of G in the form $g=hk$, $g \in G$, $h \in H$, $k \in K$, is unique if and only if the intersection of H and K, $D=H \cap K=1$, the unit-element. Since $HK=KH$, there is in this case also a unique representation in the form $g=k'h'$, and, consequently, a permutation $h \to h'$ associated with a given factorisation. Some authors deal mainly with this permutational aspect of the situation, others aim at obtaining information on the structure of G, if that of H and K is known. The outstanding result in this direction is Burnside's celebrated theorem on groups of order $p^{a}q^{\beta}$ which asserts that a group which is factorisable by two prime-power groups is soluble. The literature preceding the present series is as follows: G. Zappa, Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, Perrella, Rome, 1942, pp. 119–125; G. Casadio, Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **2**, 348–360 (1941); J. Szép, Comment. Math. Helv. **22**, 31–33 (1948) and Acta Sci. Math. Szeged **12**, Pars A, 57–61 (1950); these Rev. **8**, 367, 251; **10**, 181, 855; **12**, 239.

In the first paper G is finite and it is assumed that $D=H \cap K \neq 1$. In the relation $hk=k'h'$, h' is now not unique, but for a fixed k and given h there are d solutions h', where d is the order of D. If h runs through H, then the associated h' need not be different, but the authors prove that the k' can be chosen so that the h' are all distinct. This leads to the main result of the paper. It is still possible to associate with each k a permutation $h \to h'$ such that $hkh'^{-1} \in K$ for each h. If the set of these permutations is denoted by $[k]$, and

the set of all permutations in all sets $[k]$ by $[K]$, then $[K]$ forms a group, $[1]$ is a normal subgroup of $[K]$, and each $[k]$ a coset. Moreover, $[K]$ is mapped homomorphically on $[K]/[1]$, the kernel of the homomorphism consisting of those k for which $[k]=[1]$. It follows now that G cannot be simple if D contains a normal proper subgroup of H or K and, in particular, if H or K is Abelian.

In the second paper it is shown that the last result remains true also if $D=1$, provided that both H and K are Abelian. (That a simple group may be factorisable with one Abelian factor is clear from the example of the alternating group A_5.) It is shown further that if the orders of these Abelian factors H and K are co-prime, then G is soluble. (In this connection one could mention Hölder's results on groups of square-free order and Burnside's on groups whose Sylow subgroups are all cyclic.)

The third paper deals with the case of cyclic groups H and K. G. Zappa [loc. cit.] had raised and partially solved the "extension" problem to give a survey over all groups G for given H and K with $G=HK$. This situation is naturally even more complex than that of Schreier's extension theory where H or K are normal in G. If now H is an infinite cyclic group and K a finite cyclic group, then the author obtains a complete classification of all non-isomorphic G in terms of simple arithmetic functions. If K is also an infinite cyclic group, then the classification is not complete, but the exceptional cases are too complicated to describe here. (The author announces that in a subsequent paper II he proposes to treat the case of two cyclic groups of finite order. In a recent series of papers J. Douglas has dealt extensively with that case [Proc. Nat. Acad. Sci. U. S. A. **37**, 604–610; 677–691; 749–760; 808–813 (1951); these Rev. **13**, 621].)

In the fourth paper the author proves that if the orders of H and K are co-prime and H and K are maximal subgroups of G, then G is simple; except when H or K are of prime order in which case either H or K is normal in G. In the fifth paper he shows that if H is a p-group and K a group with centre, then G cannot be simple. This leads to the following corrollary: If H is a p-group and K Abelian, and the orders of H and K co-prime, then G is soluble.

This result appears as a very special case of the two important theorems of the sixth paper which mark a great step forward: If H is a nilpotent group, and K an Abelian group or a p-group, then G is soluble. The proofs are short but intricate. (These results of N. Itô have been used recently by H. Wielandt to prove that the product of any number of permutable groups of which one is nilpotent and the others Abelian or p-groups is soluble [Math. Z. **55**, 1–7 (1951); these Rev. **13**, 621].)

The seventh paper gives a criterion for simplicity of factorisable groups. G is simple if and only if it possesses no factorisation in which the intersection $D=H\cap K$ contains a proper normal subgroup of H or K [see the first paper above]. The only exception is that one factor is maximal, of constant order, simple, and not factorisable. (It is unlikely that this situation can be realised except in groups of order pq.) *K. A. Hirsch* (London).

Gruenberg, K. W. A note on a theorem of Burnside. Proc. Cambridge Philos. Soc. **48**, 202 (1952).

If a finite group G is factored in the form $G=HK$, where H is nilpotent and K a p-group, then G is soluble. This result has also been obtained by N. Itô [see Theorem II of the sixth paper in the preceding review]. The author's proof is essentially the same. *K. A. Hirsch* (London).

Itô, Noboru. On a theorem of L. Rédei and J. Szép concerning p-groups. Acta Sci. Math. Szeged **14**, 186–187 (1952).

Rédei and Szép [Monatsh. Math. **55**, 200–205 (1951); these Rev. **13**, 203] have recently proved the following result: Let G be a p-group, H a subgroup, a an element of G, and $G=\{H, a\}$. Denote the derived group of X by $D(X)$. Theorem: If $D(G)$ contains $D(\{H, ap^2\})$ properly, then $D(G)$ also contains $D(\{H, a^p\})$ properly. The author generalizes this in the following manner. Let G be a p-group, F the Frattini subgroup, and H any subgroup of G. If $D(G)$ contains $D(H)$ properly, then $D(G)$ also contains $D(FH)$ properly. Rédei and Szép had made the following two conjectures: (1) The index of $\{H, a^p\}$ in G is greater than that of $\{H, a^{p^2}\}$ in $\{H, a^p\}$. (2) The index of $D(\{H, a^p\})$ in $D(G)$ is greater than that of $D(\{H, a^{p^2}\})$ in $D(\{H, a^p\})$. The author disproves both conjectures by means of counterexamples. *K. A. Hirsch* (London).

Itô, Noboru. Remarks on O. Grün's paper "Beiträge zur Gruppentheorie. III." Math. Nachr. **6**, 319–325 (1952).

A knowledge of the paper quoted in the title [Math. Nachr. **1**, 1–24 (1948); these Rev. **10**, 504] is assumed, but the relevant definitions of p-normality, p-regularity, and p-hyperregularity of a finite group are repeated. The author proves the following results. (1) Not every finite group is p-regular. (In fact the alternating group A_{p^3} for odd p is not p-regular, the A_8 not 2-regular.) (2) Not every p-regular group is p-hyperregular. (Let p be an odd prime and q a prime such that $q^m\equiv 1 \pmod{p^m}$ with $m\geqq 2$. Then the special linear group $SL(p, q^n)$ is p-normal, but the linear fractional group $PSL(p, q^n)$ not p-regular. Hence $SL(p, q^n)$ is not p-hyperregular.) (3) In any soluble group G the weak closure of the centre of a p-Sylow subgroup P (i.e., the join of all conjugate subgroups of the centre of P which are contained in P) is Abelian. (4) There exist soluble groups which are not p-regular (the ingenious construction of an example uses a heavy apparatus of results on modular representations) and soluble groups which are p-regular but not p-hyperregular. (5) However, a soluble group with an Abelian derived group is p-normal and p-hyperregular for every p, and one with an Abelian second derived group is p-hyperregular for every p. *K. A. Hirsch* (London).

Smirnov, D. M. On automorphisms of solvable groups. Doklady Akad. Nauk SSSR (N.S.) **84**, 891–894 (1952). (Russian)

Mal'cev [same Doklady **67**, 23–25 (1949); these Rev. **11**, 78] has introduced the classes of A_3-groups and of A_4-groups. Every periodic group of automorphisms of a solvable A_4-group is finite. Every periodic group of automorphisms of a solvable A_3-group is a finite extension of an abelian A_3-group. *R. A. Good* (College Park, Md.).

Suprunenko, D. Soluble groups of matrices. Doklady Akad. Nauk SSSR (N.S.) **83**, 183–186 (1952). (Russian)

The author continues his study of maximal irreducible soluble subgroups of the general linear group $GL(n, F)$ over an arbitrary field. In the case of a finite ground field we have the classical problem posed by Galois (construction of primitive permutation groups of given degree) to which the author has successfully contributed [Mat. Sbornik N.S. **20(62)**, 331–350 (1947); these Rev. **8**, 562]. The case of an arbitrary ground field has first been treated in an earlier paper by the author [Učenye Zapiski Byeloruss. Gos. Univ. Ser. Fiz.-Mat. **12** (1951)]. Unfortunately this publication is

not available and the reviewer had to reconstruct its contents. Apparently the author has described a construction for maximal Abelian normal subgroups of irreducible maximal soluble subgroups and has proved that a maximal Abelian normal subgroup of an irreducible soluble subgroup of $GL(n, F)$ is the direct product of s copies of the multiplicative group of a field K of degree m over the ground field F, where moreover ms divides n. The main results of the present paper are the following two theorems whose proofs depend heavily on results obtained in the previous paper: A maximal irreducible soluble subgroup of the group $GL(n, F)$ with an infinite ground field F possesses a unique maximal Abelian normal subgroup. A maximal soluble subgroup of the group $GL(n, F)$ over an algebraically closed field splits into only a finite number of classes of conjugate subgroups. If the assumption on algebraic closure is omitted, the theorem is no longer true as the example of the rationals shows. Soluble groups of matrices over an algebraically closed field have also been studied recently by A. I. Mal'cev [same Doklady **67**, 23–25 (1949); Mat. Sbornik N.S. **28(70)**, 567–588 (1951); these Rev. **11**, 78; **13**, 203].

K. A. Hirsch (London).

Sesekin, N. F. On the theory of locally nilpotent groups without torsion. Doklady Akad. Nauk SSSR (N.S.) **84**, 225–228 (1952). (Russian)

The results announced (but not proved) in this paper extend recent work by Mal'cev [Mat. Sbornik **28(70)**, 567–588 (1951); these Rev. **13**, 203; see also same Doklady **67**, 23–25 (1949); these Rev. **11**, 78]. Mal'cev had proved that a torsion-free group containing a nilpotent normal subgroup whose Abelian subgroups are all of finite rank is itself of finite rank. The author's main result is that the existence of one maximal Abelian subgroup of finite rank is sufficient to ensure that the group itself has finite rank. Theorem 1. If the subgroup H of finite rank r in an R-group G possesses an ascending central series with respect to G, then the rank of $G/\mathfrak{z}(H)=\frac{1}{2}r(r-1)$. (Here $\mathfrak{z}(H)$ is the centraliser of H in G, the rank concept is that of Černikov [ibid. **70**, 965–968 (1950); these Rev. **11**, 496], R-groups are in the sense of Kontorovič [Mat. Sbornik **22(64)**, 79–100 (1948); these Rev. **9**, 493; see also same Doklady **59**, 213–216 (1948); these Rev. **9**, 409], and the ascending central series of H with respect to G as defined by Baer [Trans. Amer. Math. Soc. **58**, 295–347 (1945); these Rev. **7**, 371]. Theorem 2. A normal subgroup N of finite rank r of a locally nilpotent group G possesses an ascending central series with respect to G, and the factor group $G/\mathfrak{z}(N)$ has finite rank not exceeding $\frac{1}{2}r(r-1)$. Theorem 3. A nilpotent group without torsion has finite rank if and only if it possesses at least one maximal Abelian subgroup of finite rank. Theorem 4. If at least one maximal Abelian subgroup of a locally nilpotent torsion-free group G has finite rank and is contained in some nilpotent normal subgroup, then G itself is nilpotent and has finite rank. The author defines an N-group as one which is covered by nilpotent normal subgroups, i.e., in which each element is contained in some proper nilpotent normal subgroup. (An N-group need not even possess a centre: infinite triangular matrices with rational coefficient and 1's along the main diagonal.) Theorem 5. In order that a torsion-free group be nilpotent of finite rank it is necessary and sufficient that it be an N-group and possess at least one maximal Abelian subgroup of finite rank. Theorem 6. If among the maximal Abelian normal subgroups of a torsion-free N-group G there is at least one of finite rank r, then G is nilpotent of finite rank not exceeding $\frac{1}{2}r(r-1)$. The last two theorems are stronger forms of results by Mal'cev [loc. cit.]. Theorem 7. If a locally nilpotent torsion-free group G with a maximal Abelian subgroup of finite rank possesses a maximal Abelian characteristic subgroup of finite rank, then G is nilpotent and has finite rank. Finally the author elucidates the structure of torsion-free groups G with maximal Abelian subgroups of rank 2, both when G is nilpotent and when G is not nilpotent but locally nilpotent.

K. A. Hirsch.

Plotkin, B. I. On the theory of solvable groups without torsion. Doklady Akad. Nauk SSSR (N.S.) **84**, 665–668 (1952). (Russian)

A group $\mathfrak{G}$ is called ρ-solvable provided it has a well-ordered series (generally of transfinite length) of normal subgroups, $E=A_0\subset A_1\subset\cdots\subset A_\alpha\subset A_{\alpha+1}\subset\cdots\subset A_\gamma=\mathfrak{G}$, such that, for each α, $A_{\alpha+1}/A_\alpha$ is an abelian group of finite rank. A group with a finite rational series is called ρ-finite. A group each of whose subgroups is an R^*-group [Plotkin, same Doklady **73**, 655–657 (1950); these Rev. **12**, 156] is called an R^{**}-group. A ρ-solvable R^*-group is the extension of its maximal normal subgroup with an ascending central series, by means of a torsion-free abelian group. A locally ρ-solvable R^{**}-group is the extension of its maximal locally nilpotent normal subgroup, by means of a torsion-free abelian group. In a locally ρ-finite ordered group every convex subgroup is normal.

R. A. Good (College Park, Md.).

Bundgaard, Svend, and Nielsen, Jakob. On normal subgroups with finite index in F-groups. Mat. Tidsskr. B. **1951**, 56–58 (1951).

An F-group is a group generated by $2p+d+r$ elements $a_1, b_1, \cdots, a_p, b_p, S_1, \cdots, S_d, c_1, \cdots, c_r$ with the defining system of relations $k_1\cdots k_p\cdot S_1\cdots S_d\cdot c_1\cdots c_r=1, S_1^{n_1}=1, \cdots, S_d^{n_d}=1$, where $k_\pi=a_\pi b_\pi a_\pi^{-1}b_\pi^{-1}$, $\pi=1, 2, \cdots, p$. The class of F-groups is, roughly speaking, the class of discontinuous groups of motions of the hyperbolic plane that are non-abelian and finitely generated. Fenchel has conjectured that any F-group possesses a normal subgroup with finite index in F and containing no element of finite order other than unity. In this note the authors prove the conjecture in the case $p>0$. As the conjecture has already been proved [Nielsen, Mat. Tidsskr. B. **1948**, 49–56; these Rev. **10**, 590] in the case $r>0$, there remains to be proved only the case $p=r=0$.

R. H. Fox (Princeton, N. J.).

Whitehead, J. H. C. On normalizators of transformation groups. J. London Math. Soc. **27**, 374–379 (1952).

Let X be any aggregate and T the group of all one-to-one transformations of X onto itself. Let $G\subset T$ be a subgroup of T; then G determines a Klein geometry in X. The subsets X', X'' of X are said to be congruent under G: $X''\equiv X'$ if there is a $g\in G$ such that $X''=gX$. If $t\in T$ is such that $tGt^{-1}\subset G$, then $tX''\equiv tX'$ is a consequence of $X''\equiv X'$. It is therefore important for the foundations of geometry whether or not for a particular geometry $tGt^{-1}\subset G$ holds for the elements t of G only. The author proves the following theorem: Let $n\geqq 1$ and let G be either (a) the real projective group in projective space P^n, (b) an elliptic group in P^n, or (c) the orthogonal group in the n-sphere S^n. Let t: $X\to X$ ($X=S^n$ or P^n) be a one-to-one transformation of X onto itself such that $tRt^{-1}\subset G$, where $R=G$ in case (a) and in case (b) if n is even, and R is the group of rotations in case (c) and in case (b) if n is odd. Moreover, let t be continuous in case (b) or (c) if $n\neq 1$. Then $t\in G$.

A. Nijenhuis.

✱**Weyl, Hermann. Symmetry.** Princeton University Press, Princeton, N. J., 1952. viii+168 pp. $3.75.

This is slightly modified version of the Louis Clark Vanuxem Lectures given at Princeton University in 1951, with two mathematical appendices. The first lecture begins by showing how the idea of bilateral symmetry has influenced painting and sculpture, especially in ancient times. This leads naturally to a discussion of "the philosophy of left and right", including such questions as the following. Is the occurrence in nature of one of the two enantiomorphous forms of an optically active substance characteristic of living matter? At what stage in the development of an embryo is the plane of symmetry determined?

The second lecture contains a neat exposition of the theory of groups of transformations, with special emphasis on the group of similarities and its subgroups: the groups of congruent transformations, of motions, of translations, of rotations, and finally the symmetry group of any given figure. Groups containing translations in only one direction are illustrated by the metamerism of "potentially infinite" plant shoots and animals with very many legs, also by the rhythm of poetry and music and by band ornaments. In the same spirit, the cyclic and dihedral groups are illustrated by snowflakes and flowers, by the animals called Medusae, and by the plans of symmetrical buildings. Similarly, the infinite cyclic group generated by a spiral similarity is illustrated by the Nautilus shell and by the arrangement of florets in a sunflower.

The third lecture gives the essential steps in the enumeration of the seventeen space-groups of two-dimensional crystallography, as first carried out by Pólya and Niggli [Z. Kristallographie **60**, 278–298 (1924)]. The author remarks that examples of all these groups occur in the decorative patterns of the ancient Egyptians, as well as in those of the Arabs [cf. Edith Müller, Thesis, University of Zürich, 1944; these Rev. **12**, 478].

The fourth lecture begins with the complete list of finite groups of congruent transformations in Euclidean 3-space, and the attenuated list produced by the crystallographic restriction. The author mentions the 230 space-groups, and Laue's discovery that the arrangement of atoms in a crystal is revealed by X-rays. In his analysis of physical space, he distinguishes between the group of "geometric automorphisms" (generated by reflections, rotations and dilatations) and the smaller group of "physical automorphisms" (congruent transformations). The latter lacks the dilatations because the elementary particles determine an absolute length. He shows how the special theory of relativity is essentially the study of the inherent symmetry of the four-dimensional space-time continuum, where the symmetry operations are the Lorentz transformations; and how the symmetry operations of an atom, according to quantum mechanics, include the permutations of its peripheral electrons. Turning from physics to mathematics, he gives an extraordinarily concise epitome of Galois theory, leading up to the statement of his guiding principle: "Whenever you have to do with a structure-endowed entity, try to determine its group of automorphisms".

In the first appendix he enumerates the finite groups of rotations in three dimensions. In the second he shows very simply how this enumeration can be extended to the finite groups of congruent transformations (including reflections and rotatory-reflections). *H. S. M. Coxeter.*

Freudenthal, Hans. Elementarteilertheorie der komplexen orthogonalen und symplektischen Gruppen. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 199–201 (1952).

Let L be the general n-dimensional linear group over an algebraically closed field, E a nonsingular symmetric or skew-symmetric matrix, G the group of A satisfying $A'EA=E$, Γ the set of A satisfying $A'E+EA=0$ and Σ the set of A satisfying $A'E=EA$. It is proved that if $A_2=S^{-1}A_1S$ for $A_1, A_2 \varepsilon G$ (or Γ, or Σ) and $S \varepsilon L$, then the same is true for S replaced by $V \varepsilon G$. The proof uses a matrix square root U of $SS^*=S(E^{-1}S'E)$ such that $UA_1=A_1U$ and then $V=U^{-1}S$; G, Γ and Σ are conveniently characterized by $AA^*=1$, $A+A^*=0$ and $A=A^*$, respectively. *W. Givens.*

Gamba, Augusto. Sui caratteri delle rappresentazioni del gruppo simmetrico. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 167–169 (1952).

This paper presents a revised version of a method of calculating the characters of the symmetric group given by Murnaghan [Proc. Nat. Acad. Sci. U. S. A. **37**, 55–58 (1951); these Rev. **12**, 587], illustrated by an example. *G. de B. Robinson* (Toronto, Ont.).

Matossi, Frank. Irreducible representations of cubic groups. J. Chem. Phys. **19**, 1612–1613 (1951); Errata: **20**, 756 (1952).

Für die Klasse T_d (hemimorphe Hemiedrie des kubischen Systems) werden einige der bekannten Matrizen in orthogonalem Koordinatensystem angegeben. *J. J. Burckhardt.*

Bauer, Friedrich L. Sur les représentations spinorielles. C. R. Acad. Sci. Paris **234**, 1743–1744 (1952).

It was shown by E. Cartan [Bull. Soc. Math. France **41**, 53–96 (1913)] that all irreducible representations of the proper orthogonal group O_n in n dimensions are obtainable as irreducible constituents of the Kronecker products $[S]^N$ of a single "fundamental" representation S, now called the "spin" representation. With only a brief indication of proof the present note states a basic new theorem which gives the frequency with which any irreducible representation of O_n occurs in $[S]^N$. Applications of the theorem are indicated to de Broglie's theory of fusion for quantum particles of higher spin and to extending the results of Michel [J. Phys. Radium **12**, 793–804 (1951); these Rev. **13**, 807] on the irreducible tensors which can be formed from four Dirac wave functions. *A. J. Coleman* (Toronto, Ont.).

Naĭmark, M. A. Description of all irreducible unitary representations of the classical groups. Doklady Akad. Nauk SSSR (N.S.) **84**, 883–886 (1952). (Russian)

It is shown that there are no continuous unitary irreducible representations of a classical Lie group other than the known ones treated by Gelfand and Naĭmark [Trudy Mat. Inst. Steklov. **36** (1950); these Rev. **13**, 722]. The present proof is by a simpler method than the earlier proof which had been carried out only for the case of the unimodular group (loc. cit.). It uses certain new algebraic results and new estimates of norms in group rings. *I. E. Segal* (Chicago, Ill.).

Kadison, Richard V. Infinite unitary groups. Trans. Amer. Math. Soc. **72**, 386–399 (1952).

An infinite unitary group, as defined here, is a group M_U of all unitary operators contained in a Murray-von Neumann factor M [Ann. of Math. **37**, 116–229 (1936)]. The

author shows that, if M is of type II_1 or III, then the only proper closed invariant subgroups of M_U are the subgroups of the center (all complex λ such that $|\lambda| = 1$). In order to state the results obtained for the other types of factors, some definitions are needed. Let C be a normal operator in the factor M and α a number in the spectrum of C. Then α is called a "center of infinite density" (relative to M) if the spectral projection of C corresponding to any open set about α has infinite relative dimension in M. Denote by $\mathfrak{G}_f$ the set of all unitary operators in M each of which has exactly one center of infinite density and denote by $\mathfrak{G}_f^{(1)}$ the set of those elements in $\mathfrak{G}_f$ which have the point 1 as their unique center of infinite density. Now, if M is a factor of type I_∞ or II_∞, then $\mathfrak{G}_f$ is a proper closed invariant subgroup of M_U, and the only proper closed invariant subgroups of M_U are closed invariant subgroups of $\mathfrak{G}_f$. Furthermore, the only proper closed invariant subgroups of $\mathfrak{G}_f$ are $\mathfrak{G}_f^{(1)}$, finite subgroups of the center, and subgroups generated by these. It is worth noting that $\mathfrak{G}_f$ is equal to the closure of the set $\mathfrak{G}_f'$ of all unitary operators in M which act as a scalar multiple of the identity on the complement of a subspace having finite relative dimension. $\mathfrak{G}_f'$ is itself an invariant subgroup of M_U. The above results assume separable Hilbert space; however, they can be extended with the usual modifications to the non-separable case. The author is pessimistic about obtaining elegant results for arbitrary rings of operators. *C. E. Rickart* (New Haven, Conn.).

Kadison, Richard V., and Singer, I. M. Some remarks on representations of connected groups. Proc. Nat. Acad. Sci. U. S. A. **38**, 419–423 (1952).

It is shown that a connected locally compact group with a separating family of (weakly) continuous unitary representations into rings of finite type (in the sense of Murray and von Neumann) is the direct product of vector and compact groups. It follows that a non-compact simple Lie group has no non-trivial representations into a finite factor, a result due to Segal and von Neumann [Ann. of Math. **52**, 509–517 (1950); these Rev. **12**, 242] whose proof was along different lines from the present one, which makes strong use of Freudenthal's characterization of maximally almost periodic groups. Among other interesting corollaries of the proof is the following result of Singer: a connected locally compact group with a separating family of uniformly continuous unitary representations is the direct product of vector and compact groups. The same results were independently obtained by H. A. Dye (withdrawn from publication because of slightly prior submission of present paper).

I. E. Segal (Chicago, Ill.).

Yamabe, Hidehiko. On a locally compact group with a neighbourhood invariant under the inner-automorphisms. Proc. Japan Acad. **27**, 55–56 (1951).

The author proves the theorem: Let G be a locally compact connected group with a fixed neighborhood U. If U is invariant under inner automorphisms, then G contains a compact normal subgroup N such that G/N is isomorphic to the direct product of a vector group and a compact group. This theorem was first proved for Lie groups by the reviewer [Ann. of Math. **54**, 339–344 (1951); these Rev. **13**, 206] and generalized to the locally compact case by Iwasawa [ibid. **54**, 345–348 (1951); these Rev. **13**, 206]. The author has found independently, the same proof as Iwasawa.

G. D. Mostow (Baltimore, Md.).

Nakamura, Masahiro, and Turumaru, Takasi. On the representations of positive definite functions and stationary functions on a topological group. Tôhoku Math. J. (2) **4**, 1–9 (1952).

Let G be a topological group (not necessarily locally compact) and let $\mathfrak{H}$ be a Hilbert space. A function $t \to x_t$ from G to $\mathfrak{H}$ is stationary if it is continuous and the scalar product $x_t \cdot x_s$ depends only upon ts^{-1}. The authors' main theorem is to the effect that, when G is separable, every stationary function is of the form $(\phi)U_t$ where ϕ is a fixed element of $\mathfrak{H}$ and $t \to U_t$ is a strongly continuous unitary representation of G. [Reviewer's remark: Only trivial modifications seem to be necessary to make the authors' proof work for nonseparable G.] As a corollary a theorem of Khintchine on the convergence of means of stationary sequences is derived from the mean ergodic theorem. The paper concludes with a discussion of Fourier series for stationary functions of the form $t \to (\varphi)U_t$ where G is Abelian. The theorems are generalizations of some results of Ky Fan for stationary sequences [Ann. of Math. **47**, 593–607 (1946); these Rev. **8**, 568].

G. W. Mackey (Cambridge, Mass.).

Nakamura, Masahiro, and Umegaki, Hisaharu. A remark on theorems of Stone and Bochner. Proc. Japan Acad. **27**, 506–507 (1951).

The theorem of M. H. Stone referred to here (as extended by Ambrose and Raikov) reads as follows: Let $g \to g^\#$ be a continuous unitary representation of the locally compact abelian group G. Then $g^\#$ has the form $g^\# = \int \chi(g) de(\chi)$, where $de(\chi)$ is a resolution of the identity defined for all Borel sets of the character group G^*. The authors give another proof of this theorem based on a representation theorem for commutative C^*-algebras due to Dunford [Acta Sci. Math. Szeged **12**, Pars B, 51–56 (1950); these Rev. **11**, 600]. They also point out that the theorem of Bochner (as generalized by Weil and Raikov) concerning representation of positive definite functions can be obtained from the Stone theorem.

C. E. Rickart (New Haven, Conn.).

Stone, M. H. On the foundations of harmonic analysis. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **21**, no. 17, 21 pp. (1952).

The author shows how harmonic analysis on locally compact Abelian groups may be developed in a concise and natural manner from his own work on the foundations of functional analysis. His treatment includes proofs of the Plancherel theorem, the Pontrjagin duality theorem, the spectral decomposition theorem for unitary representations, the generalized Bochner theorem on the representation of positive definite functions and the Wiener general Tauberian theorem. The reviewer knows of no other place in the literature where all of these theorems are given a single unified presentation. Except in the proof of the Tauberian theorem, where appeal is made to the Gelfand-Mazur theorem, no nonelementary tools are used except the author's generalization of the Weierstrass approximation theorem, his characterization of the space of continuous functions, and his recent treatment of the Daniell-Lebesgue integral.

G. W. Mackey (Cambridge, Mass.).

Smith, P. A. The complex of a group relative to a set of generators. I. Ann. of Math. (2) **54**, 371–402 (1951).
Smith, P. A. The complex of a group relative to a set of generators. II. Ann. of Math. (2) **54**, 403–424 (1951).
★**Smith, P. A. Some topological notions connected with a set of generators.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 436–441. Amer. Math. Soc., Providence, R. I., 1952.

Two different notions are treated under a common aspect: the totality of paths of a complex and the topological group nucleus. Both are groupoids in a sense which requires no specification in this review. The main tools are the first and the second homotopy groups $\mathfrak{p}_1$ and $\mathfrak{p}_2$ of the whole groupoid G with respect to a subset V; in the case of the paths groupoid, V consists of closed paths, in the case of a topological group G, V will be a symmetric nucleus; whenever only a germ X is given, G will be the group freely generated by X, whereas V will consist of the local relators from X.

$\mathfrak{p}_1$ is defined as the factor group of G with respect to the normal divisor generated by V. In order to define $\mathfrak{p}_2$ the elements of this normal divisor are written in a formal manner:

$$(*) \qquad (g_1 * v_1)\cdots(g_n * v_n) \quad (g_i \in G, v_i \in V).$$

After having introduced some evident equivalence relations, we can freely generate a group $\mathfrak{A}$ by means of the expressions (*). The operation β carries (*) into $g_1 v_1 g_1^{-1}\cdots g_n v_n g_n^{-1}$. The kernel of β is of course $\mathfrak{p}_2$. In the case of the paths groupoid of a complex K, $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are exactly the first homotopy groups of K. In the case of a group Q and its nucleus X, the same holds if K is defined as follows: a finite set of elements of Q is a simplex if the quotient of each pair belongs to X.

$\mathfrak{p}_1$ is important when studying coverings. $\mathfrak{p}_2$, however, is related to the extendibility of homomorphisms. A homomorphism of a group germ E onto a nucleus Y of a group Q is called extendible over Q when it is contained in a homomorphism of a bigger group onto Q. If Q is simply connected and Y does not contain any element of order 2, a necessary and sufficient condition for extendibility is the triviality of the second homotopy group of Y.

This appears to be the abstract background of the Cartan theorem on the global existence of Lie groups with a given germ. The triviality of the second homology group of a simply connected Lie group is the topological tool of one of the Cartan proofs. In the proof sketched in the third paper, the triviality of $\mathfrak{p}_2$ for semisimple groups plays an analogous (but because of the abstractness more perspicuous) role. All considerations are somewhat complicated by the necessity of dealing with two germs rather than one, in order to formulate the extension theorems. *H. Freudenthal.*

Mostow, George Daniel. The extensibility of local Lie groups of transformations and groups on surfaces. Ann. of Math. (2) **52**, 606–636 (1950).

This paper studies Lie groups which are transitive on two-dimensional manifolds and completely classifies all groups of this kind as well as the manner in which they act. Lie and others have studied this problem locally and one question which arises is when a local group transitive on a local space can be extended together with the local space so as to obtain a transformation group in the large. This extensibility question is related to the question of when a subgroup of a local group determines a closed subgroup of the associated simply connected group in the large. It is shown that an $(r-k)$-parameter subgroup of an r-parameter simply connected Lie group is closed if $k<5$. If $k\geqq 5$, this need not be true. It is also proved that any semi-simple subgroup of a simply connected Lie group or a compact Lie group is closed. It follows that a local Lie group of transformations defined in a neighborhood of euclidean space of dimension $\leqq 4$ can be extended to a global group on some manifold. The two-dimensional manifolds admitting a transitive Lie group are the plane, cylinder, torus, sphere, projective plane, Möbius strip, and Klein bottle. The author then gives all groups on these spaces. *D. Montgomery.*

Tôgô, Shigeaki. On the extension of semi-simple subnuclei in Lie groups. J. Sci. Hiroshima Univ. Ser. A. **14**, 97–99 (1950).

Let H' be a local Lie subgroup in a Lie group G. In general there exists no Lie subgroup H of G with H' as its nucleus. If such an H exists then H' is said to be extensible to H. This paper investigates under what conditions a semisimple H' is extensible to a Lie subgroup H in G. It is shown that the following conditions and others are sufficient: (1) The center of G is discrete; (2) the center of H, the group generated by H', is finite; (3) the fundamental group of G is finite; (4) G is semisimple; (5) H' is complex semisimple; (6) G is a complex Lie group. Some of these conditions were also obtained by Mostow [see the preceding review]. *D. Montgomery* (Princeton, N. J.).

Calabi, Lorenzo. Sulla dimensione dei sottogruppi non chiusi di un gruppo di Lie. Boll. Un. Mat. Ital. (3) **6**, 206–208 (1951)

Let G be a simply connected Lie group. The theorem proved gives, in terms of the ranks of certain compact subgroups of G, a necessary and sufficient condition that G admit a non-closed Lie subgroup H. An upper bound given for dim H sharpens the value dim $G-5$ obtained by Mostow [see the second preceding review]. *P. A. Smith.*

Numakura, Katsumi. On bicompact semigroups. Math. J. Okayama Univ. **1**, 99–108 (1952).

Let S be a bicompact, Hausdorff, topological semigroup, that is, S is a semigroup and a bicompact Hausdorff space in which the mapping $(x, y)\to xy$ from $S\times S$ to S is continuous. If S satisfies a two-sided cancellation law, it is a group. Any bicompact (T_1) semigroup has a unique minimal two-sided ideal K (called the kernel after Suschkewitsch [Math. Ann. **99**, 30–50 (1928)]) which is bicompact and completely simple in the sense of D. Rees [Proc. Cambridge Philos. Soc. **36**, 387–400 (1940); these Rev. **2**, 127]. If S has an element 0 such that $0x=x0=0$, for all x in S, then $K=0$. The kernel K contains at least one idempotent and every minimal left (right) ideal is of the form Ke (eK), where e is an idempotent in K. The kernel is the join of all the minimal left (right) ideals of S. It can also be written as the join of disjoint, isomorphic groups. Many of the lemmas are of interest for their own sake and some proofs depend on the above cited papers. *M. Henriksen* (Lafayette, Ind.).

Moriya, Mikao. Zur Theorie der halb-topologischen Gruppen und Körper. Math. J. Okayama Univ. **1**, 109–124 (1952).

A semi-topological group G is a group that is a topological semigroup [see the review above; some proofs depend on this paper]. If G is T_1 and bicompact, it is a topological group. If G is Hausdorff, locally compact, and totally dis-

connected (diskontinuierlich) or locally connected, it is a topological group. A semi-topological division ring K is a division ring whose additive and multiplicative groups are semi-topological. It is shown that if K is T_1 and locally compact, then it is Hausdorff and satisfies the first countability axiom. Such a K is either totally disconnected or locally connected. It then follows that K is a topological division ring. This extends results of Jacobson and Taussky [Proc. Nat. Acad. Sci. U. S. A. **21**, 106–108 (1935)] and Otobe [Jap. J. Math. **19**, 189–202 (1945); these Rev. **7**, 237]. No examples of semi-topological groups (division rings) not topological groups (division rings) are cited in the paper. (An example of such a group is given by R. Arens [Bull. Amer. Math. Soc. **53**, 623–630 (1947); these Rev. **9**, 6].) *M. Henriksen* (Lafayette, Ind.).

Krull, Wolfgang. Halbgeordnete Gruppen und asymptotische Grössenordnung. Arch. Math. **3**, 1–7 (1952).

For a given subgroup S of a partly ordered Abelian group A, let S^* be the set of all elements x of A, such that $x+s\geqq 0$ and $x+s'\leqq 0$ for suitable $s, s'\in S$. Then $S^{**}=S^*$. Further, if we define $a+S^*>0$ to mean that $a+s>0$ for some $s\in S^*$, while $a+s'<0$ for no $s'\in S^*$, and $a\gg 0$ to mean that $a+s>0$ for all $s\in S^*$, then we get two partial orderings of the group A/S^*. The preceding concepts are applied to the asymptotic behavior of eventually positive functions. In the group of all such functions, S may be taken as the set of all functions eventually one, or as the set of all functions having 1 for their limit. In particular, $f(x)=O(g(x))$ may be defined in this way. *G. Birkhoff* (Cambridge, Mass.).

Trevisan, Giorgio. Sulla equivalenza archimedea relativa alle gruppo-strutture. Rend. Sem. Mat. Univ. Padova **20**, 425–429 (1951).

On sait [voir l'analyse de F. Loonstra, Compositio Math. **9**, 130–140 (1951); ces Rev. **13**, 625] que si G est un l-groupe abélien et I la relation de préordre définie par $(a, b)\in I \rightleftarrows$ "il existe un entier n tel que $|b|\leqq n|a|$", alors $I/I\cap I^{-1}$ est un ordre latticiel distributif. L'auteur montre qu'à condition de modifier très légèrement la définition de I, ceci reste vrai quand G est un l-groupe quelconque. *J. Riguet.*

Michiura, Tadashi. Sur les groupes ordonnés. II. C. R. Acad. Sci. Paris **234**, 1422–1423 (1952).

Michiura, Tadashi. Sur les groupes ordonnés. III. C. R. Acad. Sci. Paris **234**, 1521–1522 (1952).

A partly ordered (additive) group G may be called "strongly archimedean" if, for any $a>0$ in G and any x in G, $na>x$ for sufficiently large n. The author shows, among other related results, that if G is a strongly archimedean partly ordered group in which $na\geqq 0$ implies $a\geqq 0$, then G is isomorphic with the lexicographic union $Q\circ V$ of a subgroup Q of a ("pseudo-cardinal") product of replicas of the additive group of reals and an unordered group V.
G. Birkhoff (Cambridge, Mass.).

NUMBER THEORY

***Gloden, A. Table de factorisation des nombres N^4+1 dans l'intervalle $3000<N\leqq 6000$.** Published by the author, Luxembourg, 1952. i+51 pp. 80 Belgian francs.

The table described in the title is an extension of a similar table of A. J. C. Cunningham for $N\leqq 1000$ [Binomial factorisations, v. 1, Hodgson, London, 1923, pp. 113–119] and a manuscript table of the author for $1000<N\leqq 3000$. Many of the entries in the present table are complete factorizations. Those which are incomplete involve unknown prime factors $>6\cdot 10^8$. *D. H. Lehmer.*

Palamà, Giuseppe. Osservazioni sul "Neocribrum" di L. Poletti. Boll. Un. Mat. Ital. (3) **7**, 63–67 (1952).

Poletti's Neocribrum is a method of laying out a factor table of numbers prime to 30 in such a way that numbers divisible by 7, 11, and 13 are indicated by the printer in advance. The present note discusses various methods of determining and checking for each column of the table the first place which corresponds to a number divisible by a given prime [see also Palamà, Rivista Mat. Univ. Parma **1**, 85–98 (1950); these Rev. **11**, 499]. *D. H. Lehmer.*

***Ξηρουδάκης, Γεώργιος Φ., καὶ Φασουλάκης, Κωνστ. Ν. Εἰσαγωγὴ εἰς τὴν διοφαντικὴν ἀνάλυσιν ἀνωτέρου βαθμοῦ. [Xeroudakes, Georgios Ph., and Fasoulakes, Konst. N. Introduction to diophantine analysis of higher degree.]** N. D. Phrantzeskakes, Athens, 1947. 148 pp. 12,000 drachmas.

This book was written with the aim of stimulating interest among young mathematicians in the direction of Diophantine Analysis. For this purpose the authors have written a very elementary book which presupposes only a bare minimum of background in the theory of numbers. No knowledge of quadratic residues or algebraic number fields is required.

The introduction is devoted to equations of the second degree. Parametric solutions are given of the three equations $x^2+ay^2=z^2$, $x^2+y^2=Az^2$, $x^2+y^2=z^2+w^2$. Then there is the usual discussion of numbers of the form $x^2+Axy+By^2$ with applications to solutions of such equations as $x^2+y^2=z^2$, $X^2+Y^2+Z^2=x^4+y^4+z^4+w^4$, $x^2+ay^2=z^2+aw^2$, $x^2-xy+y^2=z^2$, $x^2+y^2+z^2=w^2$. This is followed by a chapter on arithmetic triangles and parallelograms. The second chapter deals with a large number of equations of the third and fourth degree, most of which are of the form $f(x^3, y^3, z^3)=g(X^3, Y^3, Z^3)$ and $f(x^4, y^4, z^4)=g(X^4, Y^4, Z^4)$. In chapter III the study of third and fourth degree equations is continued, the principal types considered being of the form $f(x, y)=mz^2$, where f is a homogeneous polynomial of degree three or four. The fourth and final chapter introduces Fermat's method of infinite descent and applies it to prove the impossibility of certain equations, for instance: $x^4+y^4=z^4$, $x^4+6x^2y^2+y^4=2z^2$, $x^4+9x^2y^2+8y^4=z^2$, and other equations derivable from these. The book also contains fifty exercises. *T. M. Apostol* (Pasadena, Calif.).

Nagell, Trygve. Bemerkung über die Diophantische Gleichung $u^3-Dv^3=C$. Arch. Math. **3**, 8–9 (1952).

Referring to previous work [Arch. Math. **2**, 161–165 (1950); these Rev. **11**, 714] the author indicates that a simpler formulation results if the fundamental solution of a class were defined as that one with least non-negative v instead of least positive v. *I. Niven* (Eugene, Ore.).

Skolem, Th. Application of 3-adic analysis and "cofields" to the proof of some theorems concerning certain cubic equations. Norsk Mat. Tidsskr. **34**, 45–51 (1952). (Norwegian)

Through the application of 3-adic expansions and the author's concept of cofields certain results by Delaunay and

Nagell concerning the integral solutions of the Diophantine equations $x^3+Dy^3=1$ and $Ax^3+By^3=1$ or 3 are derived in a very simple manner. *O. Ore* (New Haven, Conn.).

Alef. On a diophantine equation. Revista Mat. Hisp.-Amer. (4) **12**, 3–8 (1952). (Spanish)

The equation $x^3+y^3=2z^2$ is found to have solutions $x, y=2^{2\alpha}3^{2\gamma+1}r^2\pm s$, $z=2^{\alpha+\beta}3^{\gamma+1}rq$, where the integral parameters satisfy the relation $2^{2\beta}q^2-s^2=2^{4\alpha}3^{4\gamma+1}r^4$. It is asserted that these include all relatively prime solutions, but an error in the interpretation of the writer's formulas [9] leads him to overlook another class of solutions. For example the solution $x=253$, $y=85$, $z=2899$ does not arise from the above formulas, which give only values with $z\equiv 0 \pmod 3$.

I. Niven (Eugene, Ore.).

Kanold, Hans-Joachim. Über ein spezielles System von zwei diophantischen Gleichungen. J. Reine Angew. Math. **189**, 243–245 (1952).

The following theorem, stemming from the author's work [same J. **186**, 25–29 (1944); these Rev. **6**, 255] on odd prime numbers, is proved; the only simultaneous solution of the equations $\sum_{i=0}^{p-1}p_1^{ia}=pp_2^b$, $\sum_{i=0}^{p-1}p_2^{ic}=pp_1^d$ with arbitrary positive integers a, b, c, d and primes $p, p_1, p_2, p_1\neq p_2$, is $p=2$, $p_1=3$, $p_2=5$, $a=2$, $b=c=d=1$. *I. Niven.*

Palamà, Giuseppe. Sulle somme di K^{mo} potenze e su di un teorema relativo alle multigrade. Boll. Un. Mat. Ital. (3) **7**, 19–29 (1952).

Let $\beta(k)$ and $\gamma(k)$ denote the smallest value of n for which the equation

$$(1)\qquad x_1^k+\cdots+x_n^k=y_1^k+\cdots+y_m^k,\quad (x_1,\cdots,x_n,y_1,\cdots,y_n)=1,$$

has one or infinitely many non-negative integral solutions respectively when $m<n$. Moreover, let $\beta'(k)$, $\gamma'(k)$ denote the smallest value of n for which (1) has one or infinitely many solutions respectively when $m<n-1$. Then $\beta(k)\leqq\gamma(k)$, $\beta(k)\leqq\beta'(k)$ and $\gamma(k)\leqq\gamma'(k)$. In the first part of this paper the author obtains some upper bounds for $\beta(k)$, $\beta'(k)$, $\gamma(k)$ and $\gamma'(k)$ for $k\leqq 24$. In the second part of the paper the author gives a theorem regarding the combination of two multigrade equations of degree n to form a multigrade equation of degree $n+1$. *W. H. Simons.*

Ore, Oystein. The general Chinese remainder theorem. Amer. Math. Monthly **59**, 365–370 (1952).

Consider the system of simultaneous congruences $x\equiv a_i \pmod{m_i}$, $1\leqq i\leqq k$. It is well known that a necessary and sufficient condition for their solvability is that $a_i\equiv a_j \pmod{d_{i,j}}$ where $d_{i,j}=(m_i, m_j)$. Put $M=[m_1, m_2, \cdots, m_k]$. Clearly x is uniquely determined (mod M). The theorem of the author states that x can be written in the form $x\equiv\sum_{i=1}^k a_ic_iM/m_i$, where the c_i form a set of integers satisfying $\sum_{i=1}^k c_iM/m_i=1$. This result can be considered as a generalization of the Chinese remainder theorem. The author proves some other related results. *P. Erdős.*

Matulewicz, K. Sur la solution d'une congruence en nombres composés. Colloquium Math. **2** (1951), 261–263 (1952).

The congruence mentioned in the title is $a^n\equiv a \pmod n$ where n is a composite integer. The author gives two known theorems about the case in which n is a product of two primes and lists all solutions n for which $2000<n\leqq 4033$ in case $a=2$. The author is unaware of the fact that all solutions of this congruence were given by P. Poulet up to $n=10^8$ [C. R. 2ième Congrès Internat. Récréation Math., Paris, 1937, Bruxelles, 1937, pp. 74–84]. [See Lehmer, Amer. Math. Monthly **56**, 300–309 (1949); these Rev. **10**, 681.] *D. H. Lehmer* (Los Angeles, Calif.).

Mihaljinec, Mirko. Une contribution au problème de Fermat. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **7**, 12–18 (1952). (Serbo-Croatian. French summary)

The author uses quite elementary methods to prove two results about the equation (1) $x^n+y^n=z^n$ $(n>2)$. (I) The equation (1) cannot be solved in integers x, y, z which are in arithmetical progression. (II) If the equation (1) has a solution for which $z-x$ is square-free, then $x<n^3$ and $y<n^3+n$, so that (1) has only a finite number of such solutions. Unfortunately, Peter Barlow [Theory of numbers, Johnson, London, 1811] proved that $z-x$ is not square-free when n is a prime, the really interesting case.

D. H. Lehmer (Los Angeles, Calif.).

Natucci, Alpinolo. Osservazioni sul problema di Fermat. Boll. Un. Mat. Ital. (3) **6**, 245–248 (1951).

Let $y=x+u$, and $z=x+u+v$. Then the function

$$z^n-(x^n+y^n)$$

becomes

$$F_n=F_n(x,u,v)=(x+u+v)^n-\{x^n+(x+u)^n\}.$$

The author observes that with x, u, v fixed and positive, $F_n\to-\infty$ as $n\to\infty$. From this fact he concludes that it is extremely improbable that F_n can be made to vanish. A small table of $F_n(x,u,v)$ for $v=1$, $u=1, 2$, and $n=3, 4, 5$ is given. The variable x ranges over integers in each case as far as the place where F becomes negative.

D. H. Lehmer (Los Angeles, Calif.).

Mycielski, Jan. Sur les représentations des nombres naturels par des puissances à base et exposant naturels. Colloquium Math. **2** (1951), 254–260 (1952).

Let n be an integer. Consider all representations $n=a_i^{b_i}$, a_i, b_i integers, $n=a_1>a_2>\cdots>a_{\gamma(n)}$, $b_1=1<b_2<\cdots<b_{\gamma(n)}$ (i.e., $\gamma(n)$ denotes the number of representations of n in the form k^l). Put $\tau(n)=\sum_1^{\gamma(n)}b_i$. The author proves, among other things, that $\sum_{n=2}^{n}(\tau(n)-1)/n=x^2/6+1$. *P. Erdős.*

Cugiani, Marco. Sulle funzioni simmetriche di particolari sistemi di interi. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **14** (**83**), 529–543 (1950).

This generalizes certain results of R. Tambs Lyche [Avh. Norske Vid. Akad. Oslo. I. **1944**, no. 9 (1945); these Rev. **7**, 505] and W. Ljunggren [Norsk Mat. Tidsskr. **27**, 101–106 (1945); Norske Vid. Selsk. Forh., Trondhjem **17**, no. 28, 110–113 (1944); these Rev. **8**, 314, 368]. For $0\leqq k\leqq(p-1)n$ define $W_k(n)$ by $\sum_{k=0}^{n(p-1)}W_k(n)x^{n(p-1)-k}=\prod(x+j)$, where the range of this product is $0<j<np$, $(j,p)=1$. Let $p^{\beta-1}$ be the highest power of p dividing $W_k(n)$, and write $s=[k/(p-1)]$. Then p^β does not divide $\binom{n}{s+1}$ if $(p-1)\nmid k$. For p odd, $W_k(n)\equiv(-1)^s\binom{n}{s} \pmod{p^\beta}$ for $k=s(p-1)$, and

$$W_k(n)\equiv\tfrac{1}{2}(-1)^s np(p-1)(s+1)\binom{n}{s+1} \pmod{p^\beta}$$

for $k=s(p-1)+1$. For $p=2$ the results are the same as those of Tambs Lyche. The proofs employ congruences of G. Ricci [Ann. Univ. Toscane (2) **14**, 177–198 (1930)].

I. Niven (Eugene, Ore.).

Brauer, Alfred, and Reynolds, R. L. On a theorem of Aubry-Thue. Canadian J. Math. 3, 367–374 (1951).

L. Aubry [Mathesis (4) 3, 33–35 (1913)] and A. Thue [Skrifter Vid. Kristiania, Mat.-Nat. Kl. no. 3 (1915)] proved that, under certain conditions on a, b, and m, the congruence $ax \equiv by \pmod{m}$ has solutions x and y such that $x^2 < m$ and $y^2 < m$. An historical account is given of certain generalizations. The principal generalization proved here is as follows: Let r, s be positive integers, $r < s$, and let $f_1, \cdots, f_s$ be positive numbers less than m such that $f_1 \cdots f_s > m^r$; then the system of r linear congruences $a_{i1}x_1 + \cdots + a_{is}x_s \equiv 0 \pmod{m}$ has a non-trivial solution such that $|x_j| < f_j$. Corollaries follow concerning the magnitude of coefficients of polynomials $\phi(x)$ and $\psi(x)$ such that $g(x)\phi(x) + h(x)\psi(x) \equiv 0 \pmod{f(x), p}$; and of elements of matrices U and V such that $AU \equiv BV \pmod{m}$. A very simple proof of the fact that a prime p is a sum of four squares follows from the solvability of $a^2 + b^2 + 1 \equiv 0 \pmod{p}$, $x \equiv az + bt$, $y \equiv bz - at \pmod{p}$, with x, y, z, and t less than $p^{\frac{1}{2}}$. As another corollary, each of the $k-1$ classes of kth power non-residues contains at least one element less than $p^{(k-1)/k}$. Finally, an extension is given to algebraic fields.

G. Pall (Chicago, Ill.).

Skolem, Th. Existence of an nth non-power-residue mod p less than $\sqrt{p}$. Norsk Mat. Tidsskr. 33, 123–126 (1951). (Norwegian)

By an analytic method and the use of a theorem of Aubrey-Thue the author shows that if p is a sufficiently large prime, n a natural integer for which there exists numbers which are not nth power residues (mod p), then there are non-residues which do not exceed $\sqrt{p}$. *O. Ore.*

Nagell, T. The least positive nth non-power-residue modulo p. Norsk Mat. Tidsskr. 34, 13 (1952). (Norwegian)

Let $p \geqq 11$ be a prime and $n \geqq 3$ an odd integer; furthermore, let x be the least positive integer which is not an nth power residue (mod p). It is shown through a simple elementary proof that $x < \sqrt{(\frac{1}{2}p)}$, thus improving a result by Skolem [see the preceding review]. *O. Ore.*

Straus, E. G. Functions periodic modulo each of a sequence of integers. Duke Math. J. 19, 379–395 (1952).

The author investigates integral-valued functions $f(x)$ of an integral variable x that are periodic modulo a set of integers m_i, not necessarily for each m_i with the same period. He distinguishes three types, according as periodicity holds for all x, or only for $x \geqq x_0$, or only for $x \geqq x_i$ where x_i depends on m_i. He discusses, in particular, whether such functions may assume prime values for all x of their type, respectively for all x of their type that are themselves prime numbers. Of particular interest seem the following two results. (1) If $f(x)$ is an integral-valued polynomial, and if $f(x)$ is a prime whenever x is, then $f(x) = x$. (2) If, for every sufficiently large prime p, $f(x)$ maps the complete system of residues (mod p) again on such a system (mod p), then $f(x) = ax + b$ where $a \neq 0$. *K. Mahler* (Manchester).

Toscano, Letterio. Su una relazione di ricorrenza triangolare. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 9, 247–254 (1950).

In this paper the author continues the study of the numbers $l^{(\lambda)}_{n+1,\,i+1}$ defined by

$$n!(xD)^n = \sum_{i=0}^{n} l^{(\lambda)}_{n+1,\,i+1} x^{n-i-\lambda+1} D^n x^{i+\lambda-1}$$

[same Rend. (5) 8, 337–350 (1949); these Rev. 11, 511]. Of particular interest are the special cases $l^{(1)}_{n+1,\,i} = A_{n,\,i}$ (introduced by Euler) and $2^{n-1} l^{(\frac{1}{2})}_{n,\,i} = H_{n,\,i}$. Tables for $A_{n,i}$ and $H_{n,i}$ for $1 \leqq n \leqq 10$ are given. The general numbers $l^{(\lambda)}_{n,i}$ are expressed in terms of the Stirling numbers of the second kind and also in terms of $A_{n,i}$. The explicit formula

$$l^{(\lambda)}_{n+1,\,i+1} = \sum_{j=0}^{i} (-1)^j \binom{n+1}{j} (\lambda + i - j)^n$$

is obtained. *L. Carlitz* (Durham, N. C.).

Duparc, H. J. A. On Carmichael numbers. Simon Stevin 29, 21–24 (1952).

The author defines a Carmichael number as a positive composite integer n for which $a^{n-1} \equiv 1 \pmod{n}$ for every integer a prime to n. The smallest such number is 561. It is known that a Carmichael number is odd, squarefree, and divisible by at least three primes. A necessary and sufficient condition that n be a Carmichael number is that $np - 1$ be divisible by $p - 1$ for all prime factors p of n. Using this fact, the author shows that if all but the largest two prime factors of a Carmichael number are specified, the last two factors can be assigned in only a finite number of ways. More specifically, if $n = p_1 p_2 \cdots p_r$ $(p_1 < p_2 < \cdots < p_r)$ and $m = n/(p_{r-1}p_r)$, then

$$p_{r-1} \leqq 1 + 2(m-1)^2, \quad p_r \leqq m(m-1)^2 + \tfrac{1}{2}(m+1).$$

This is a generalization of a theorem of Beeger [Scripta Math. 16, 133–135 (1950); these Rev. 12, 159]. A further argument is used to show that

$$p_{r-1} \leqq 1 + (m-1)(2m + \tfrac{1}{2} - \{m - 3/4\}^{\frac{1}{2}}).$$

Thus for $r = 3$, $m = 3$, we find $p_2 \leqq 3361$. In fact $43 \cdot 3361 \cdot 3907$ is a Carmichael number. Various other conditions are used to show that, for example, 62745 is the only Carmichael number of the form $15pq$, and that no Carmichael number is of the form $21pq$ or $39pq$. *D. H. Lehmer.*

Overholtzer, Gordon. Sum functions in elementary p-adic analysis. Amer. J. Math. 74, 332–346 (1952).

The Schur derivatives Δa_n of a sequence (a_n) are defined by $\Delta a_n = (a_{n+1} - a_n)/p^{n+1}$. Generalizing earlier results, the author considers the Schur derivatives of the sequence

$$D[i_0(p^s), p^n, g] = [i \equiv i_0 \pmod{p^s},\ 0 \leqq i \leqq p^n - 1] \textstyle\sum_i g(i) p^n$$

when $g(x)$ can be expanded into a power series

$$\textstyle\sum g^{(j)}(i_0)(x - i_0)^j / j!$$

which is convergent for $|x - i_0| \leqq |p^s|$ for p-adic valuation. It is shown that the mth Schur derivative of the mentioned sequence has the form

$$\sum_{\gamma=m}^{\infty} L^m(\gamma) p^{n(\gamma-m)} D[i_0(p^s), p^n, g^{(\gamma)}]$$

where $g^{(\gamma)}$ is the γth derivative of g and the $L^m(\gamma)$ are independent of n and g. The limit of $D[i_0(p^s), p^n, g]$ when n tends to infinity can be expressed with the aid of Bernoullian numbers and the coefficients a_k in the expansion $g(i_0 + p^s y) = \sum_{k=0}^{\infty} a_k y^k$. Applications are given to some elementary functions g. Further, the continuity and analyticity of the sum functions are examined. *H. Bergström.*

Shapiro, George. On the Dirichlet series associated with Ramanujan's τ-function. Amer. J. Math. 74, 401–409 (1952).

Let p_k denote the kth prime. The author proves that, for every fixed $\sigma = \Re s > 13/20$, the sequences $\{Z_n(s)\}$ and

$\{\text{Log } Z_n(s)\}$ tend in relative measure to $Z(s)$ and Log $Z(s)$. Here

$$Z_n(s)=\prod_{k=1}^{n}\{1-\tau(p_k)p_k^{-11/2-s}+p_k^{-2s}\}^{-1},$$

$$Z(s)=\prod_{k=1}^{\infty}\{1-\tau(p_k)p_k^{-11/2-s}+p_k^{-2s}\}^{-1}=\sum_{n=1}^{\infty}\tau(n)n^{-11/2-s},$$

$\tau(n)$ is Ramanujan's function, and Log Z denotes a particular branch of the logarithm. The infinite products and series converge absolutely if $\sigma>1$. A method similar to that used by H. Bohr [Acta Math. **40**, 67–100 (1915)] is used. The asymptotic distribution functions of $Z(s)$ and Log $Z(s)$ are also investigated by methods similar to those used by Jessen and Wintner [Trans. Amer. Math. Soc. **38**, 48–88 (1935)]. It is shown that both asymptotic distribution functions are absolutely continuous; their densities $D_\sigma(x)$, $\bar{D}_\sigma(x)$ possess continuous partial derivatives of all orders. Also, for any fixed $\lambda>0$, $D_\sigma(x)=O(e^{-\lambda|x|^2})$, $\bar{D}_\sigma(x)=O(e^{-\lambda\log^2|x|})$ as $|x|\to\infty$. Further, if $13/20<\sigma<1$, $D_\sigma(x)>0$ for all x and $\bar{D}_\sigma(x)=0$ if and only if $x=0$. Other properties are also proved. The lower bound of the abscissa 13/20 depends upon the reviewer's estimate, $\tau(n)=O(n^{29/5})$; presumably it can be replaced by 5/8 by using the estimate $\tau(n)=O(n^{22/4+\epsilon})$ which may be obtained by using A. Weil's results on Kloosterman sums [Proc. Nat. Acad. Sci. U. S. A. **34**, 204–207 (1948); these Rev. **10**, 234]. *R. A. Rankin* (Birmingham).

Carlitz, L. Note on an arithmetic function. Amer. Math. Monthly **59**, 386–387 (1952).

The author extends certain known properties of the Gaussian number-theoretical function $F(a,m)=\sum_{r=m}\mu(r)a^s$ where $\mu(r)$ is the Möbius function. Let $\phi(M)$ denote the Euler function of a polynomial M with coefficients in a Galois field $GF(p^n)$ and let A be some other such polynomial. Then for the proper divisors D of A^m-1 one has that m divides $\phi(D)$ and $F(a,m)=\sum_D\Phi(D)$ where $a=|A|=p^{n\deg A}$. *O. Ore* (New Haven, Conn).

Vinogradov, I. M. New approach to the estimation of a sum of values of $\chi(p+k)$. Izvestiya Akad. Nauk SSSR. Ser. Mat. **16**, 197–210 (1952). (Russian)

Let q be a prime, let χ be a non-principal character (mod q), and let k be a fixed integer $\not\equiv 0 \pmod q$. The sum under consideration is $S=\sum_{p\le N}\chi(p+k)$, where p runs through primes. The main result proved is that $S=O(N^{1+\epsilon}(q^{3/4}N^{-1})^{1/4})$ for any fixed $\epsilon>0$, provided q is large and N lies between fixed multiples of $q^{3/4}$ and $q^{5/4}$. This result is an improved form of that announced, with an indication of the method of proof, in a previous note [Doklady Akad. Nauk SSSR **73**, 635–638 (1950); these Rev. **12**, 161]. The proof, now given in detail, differs somewhat from that outlined earlier. One of the main ideas is still that of exhausting the hyperbolic region of summation of a double sum by rectangular regions. But the proof also uses (1) an argument similar to that used in the proof of Theorem 3 of Chapter 9 of the author's monograph [Trudy Mat. Inst. Steklov. **23** (1947); these Rev. **10**, 599], based on Lemma 5 of the same chapter; (2) a special case of A. Weil's general theorem on exponential sums [Proc. Nat. Acad. Sci. U. S. A. **34**, 204–207 (1948); these Rev. **10**, 234], namely the estimate $O(q^{1/2})$ for the generalized Kloosterman sum

$$\sum_{x=0}^{q-1}\chi\left(\frac{x+a}{x+b}\right)e^{2\pi imx/q}.$$

H. Davenport (London).

Flanders, Harley. A remark on Kronecker's theorem on forms. Proc. Amer. Math. Soc. **3**, 197 (1952).

The Kronecker theorem on forms over an integral domain is a consequence of integral closure. The author remarks that the converse of this theorem holds, which is almost trivial. *H. Bergström* (Gothenburg).

***Venkov, B. A. On indeterminate quadratic forms with integral coefficients.** Trudy Mat. Inst. Steklov., v. 38, pp. 30–41. Izdat. Akad. Nauk SSSR, Moscow, 1951. (Russian) 20 rubles.

Let $f(X)=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}x_ix_j$ be an indefinite quadratic form with integral coefficients consisting of one positive square and $n-1$ negative ones. In n-dimensional X-space, denote by $\mathfrak{R}$ one of the two convex cones $f(X)\gtrless 0$, by $\mathfrak{M}$ the set of all lattice points in the interior of $\mathfrak{R}$, and by Π the convex cover of $\mathfrak{M}$; this is an unbounded polyhedron. The author studies the fundamental regions of the group G of all automorphisms of f with integral coefficients and determinant ∓ 1, and the properties of Π with respect to this group; e.g., the $(n-1)$-dimensional sides of Π are equivalent to a finite number amongst them. *K. Mahler* (Manchester).

Bambah, R. P., and Roth, K. F. A note on lattice coverings. J. Indian Math. Soc. (N.S.) **16**, 7–12 (1952).

A lattice Λ of determinant $d(\Lambda)\neq 0$ is called a covering lattice for a symmetrical convex body K in n-dimensional space if every point of the space belongs to a body of the type $K+X$ where X is a point of Λ. The density $\theta(K)$ of the most economical lattice covering of space by K is defined to be the lower bound of $V(K)/d(\Lambda)$ for all covering lattices Λ for K; here $V(K)$ is the content of K. Hlawka has shown that $\theta(K)\le n^n$ and this has been improved by Rogers [J. London Math. Soc. **25**, 327–331 (1950); these Rev. **13**, 323] to $\theta(K)\le 3^n$. The authors use the Brunn-Minkowski theorem to construct an n-dimensional space-filling body and deduce that $\theta(K)\le \pi n^n/(3\sqrt{3}n!)$ when K is symmetrical about its coordinate planes. This is an improvement on Rogers's inequality since the expression on the right is asymptotically equal to $e^n(\pi/54n)^{\frac{1}{2}}$ as $n\to\infty$. *R. A. Rankin.*

***Schwarz, Štefan. Algebraické čísla. [Algebraic numbers.]** Jednota Československých Matematikov a Fysikov, Prague, 1950. 291 pp. (Slovak) Kčs. 136.

I. The arithmetic of rational whole numbers. II. The arithmetic of polynomials over a field. III. Algebraic numbers and algebraic number fields. IV. General theorems on integers in quadratic fields. V. Some special quadratic fields. VI. Considerations leading to the concept of ideal. VII. The arithmetic of quadratic ideals. VIII. Detailed study of quadratic fields. IX. The arithmetic of integers in algebraic fields of the nth degree. X. Fermat's problem. XI. Quaternions. *Table of contents.*

van de Vooren-van Veen, J. F. On prime numbers and prime ideals. Simon Stevin **29**, 13–20 (1952). (Dutch)

The author gives proofs of the following theorems, the proofs being very similar to those given in Hardy and Wright [An introduction to the theory of numbers, Oxford, 1938, pp. 218–221] except that a and b are determined explicitly, using Fermat's theorem: every prime $p\equiv 1 \pmod 4$ is of the form a^2+b^2; every $p\equiv 1 \pmod 6$ is of the form a^2-ab+b^2; the primes of $k(i)$ are the associates of $1+i$, $a+bi$ where $p=4n+1=a^2+b^2$, and $p=4n+3$; the primes of $k(\rho)$ are the associates of $1-\rho$, $a+b\rho$ where

$p=6n+1=a^2-ab+b^2$, and $p=3n+2$. The constructive proof of the representability of the primes in question is shorter, but more sophisticated, than that contained in a recent paper by M. v. Vlaardingen [Simon Stevin **28**, 55–59 (1951); these Rev. **12**, 676]. *W. J. LeVeque.*

Krull, Wolfgang. Zur Arithmetik der endlichen diskreten Hauptordnungen. J. Reine Angew. Math. **189**, 118–128 (1951).

Let $\mathfrak{K}$ be a field. Let D be the free abelian group generated by a set (of arbitrary cardinal) of marks P called prime divisors, call $\prod P^{\nu(P)}$ integral if all $\nu(P)\geqq 0$, call the elements of D divisors, and say that one divisor divides another if the quotient is integral. Following Hasse [Zahlentheorie, Akademie-Verlag, Berlin, 1949; these Rev. **11**, 580], the author assumes a homomorphism $\alpha\to d(\alpha)$ of the multiplicative group of $\mathfrak{K}$ into D such that: (1) if $d\mid\alpha$ (i.e., $d\mid d(\alpha)$) and $d\mid\beta$ then $d\mid(\alpha+\beta)$ ($d\mid 0$ for every $d\in D$); (2) if P, P' are prime divisors then there is an $\alpha\in\mathfrak{K}$ with $P\mid\alpha$, $P^2\nmid\alpha$, $P'\nmid\alpha$. Then the subring $\mathfrak{R}$ of all elements whose divisors are integral has $\mathfrak{K}$ as its field of quotients and is integrally closed in $\mathfrak{K}$. There is an obvious correspondence between elements of D and a subset (called the D-ideals) of the (fractional) $\mathfrak{R}$-ideals. If the sum of every two D-ideals is a D-ideal then R is a Noetherian ring. The author is interested in finding, for non-Noetherian rings, necessary and sufficient conditions that the (ordinary ideal-theoretic) product of every two D-ideals shall be again a D-ideal, and finds the following ones. (A) For every ideal $\mathfrak{p}$ corresponding to a prime divisor, $\mathfrak{p}\mathfrak{p}^{-1}=\mathfrak{R}$. (B) The primitive ideal of each class of D-ideals is $\mathfrak{R}$, where ideal class is defined in the usual way and its primitive ideal is the sum of all integral ideals in it. (C) All divisor-ideals become principal in every valuation-ring $\mathfrak{R}_\mathfrak{q}$ defined by a prime ideal $\mathfrak{q}$ in $\mathfrak{R}$.

G. Whaples (Bloomington, Ind.).

Carlitz, L. Diophantine approximation in fields of characteristic p. Trans. Amer. Math. Soc. **72**, 187–208 (1952).

Let the Galois field $GF(p^n)$ for fixed p and n be given and let $\Phi=GF[p^n, x]$ denote the field of all expressions

$$(1)\qquad \alpha=\sum_{i=m}^{\infty} c_i x^i \quad (c_i\in GF(p^n)),$$

where x is an indeterminate. We may suppose c_m to be $\neq 0$; in that case $m\gtreqless 0$ is called the degree of α ($\deg\alpha$). If $m\geqq 0$ the part $\sum_0^\infty$ of (1) is called the integer part $[\alpha]$ of α; if $m<0$, we put $[\alpha]=0$; further the part $\sum_m^{-1}$ is called the fractional part $\{\alpha\}$ of α (if $m<-1$, then $c_i=0$ for $m+1\leqq i\leqq -1$). The author investigates Diophantine approximations to the numbers $\alpha\in\Phi$. First, an analogue of Kronecker's theorem is proved which was previously shown by Mahler [Ann. of Math. **42**, 488–522 (1941); these Rev. **2**, 350]. Further, uniform distribution is defined: Let a sequence $\alpha_1, \alpha_2, \cdots$ in Φ and an arbitrary number $\beta\in\Phi$ be given. Let for fixed $N\geqq 1$ the number $N_k=N_k(N,\beta)$ of those α_i ($1\leqq i\leqq N$) be considered such that $\deg\{\alpha_i-\beta\}<-k$; we then say that the sequence $\alpha_1, \alpha_2, \cdots$ is uniformly distributed (mod 1) if $N_k/N\to p^{-nk}$, as $N\to\infty$ for any fixed couple $k\geqq 1$ and $\beta\in\Phi$.

Finally, the function $e(\alpha)$ is defined on Φ: Let θ define $GF(p^n)$; then for c_{-1} in (1) put

$$c_{-1}=\gamma_1\theta^{n-1}+\gamma_2\theta^{n-2}+\cdots+\gamma_n \quad (\gamma_i\in GF(p))$$

and $e(\alpha)=e^{2\pi i\gamma_1/p}$. Then $e(\alpha)=e(\beta)$ for $\alpha=\beta$ (mod 1), i.e., for $\alpha=\beta+A$, where A is an integer in Φ, i.e., where A is a polynomial in $GF[p^n, x]$. The author then deduces the analogue to Weyl's well-known criterion and gives applications, mainly to polynomials $\varphi(u)$ of degree $k<p$ with coefficients in Φ for which u runs through the integers $A\in\Phi$.

J. F. Koksma (Amsterdam).

Eggleston, H. G. Sets of fractional dimensions which occur in some problems of number theory. Proc. London Math. Soc. (2) **54**, 42–93 (1952).

The sets mentioned in the title occur in certain problems of diophantine approximation. If the set of real numbers θ for which a certain law of diophantine approximation holds, is a null-set, one may ask for its dimension in the sense of Hausdorff. In this domain Besicovitch, Jarník, Knichal [references in the reviewer's Diophantische Approximationen, Springer, Berlin, 1936] and Lock [Dissertation, 1947; these Rev. **9**, 79] proved several results concerning the decimal expansion of real numbers. In the first part of this paper the author deduces a number of general theorems on Hausdorff dimension, using mainly a method of Besicovitch. In the second part applications to the problems of rational approximation to irrational numbers are given, whereas part 3 contains questions concerning the distribution of digits in decimals. In part 4 among other results is proved: If a strictly increasing sequence $M_1<M_2<\cdots$ of integers is such that $M_{i+1}/M_i\to\infty$ as $i\to\infty$, then the set S_α of θ for which $M_i\theta\to\alpha$ (mod 1) ($0\leqq\theta\leqq 1$) as $i\to\infty$ has dimension 1 for any given α ($0\leqq\alpha\leqq 1$). This is in striking contrast to the case that M_{i+1}/M_i is bounded, as then S_α is at most enumerable for any α ($0\leqq\alpha\leqq 1$).

J. F. Koksma (Amsterdam).

Kuipers, L., and Meulenbeld, B. Some properties of continued fractions. Acta Math. **87**, 1–12 (1952).

The authors consider the diophantine inequality

$$(1)\qquad |\alpha-p/q|<k/q^2,$$

where α is irrational and p/q belongs to one of the three classes, odd/odd, odd/even, even/odd, of irreducible fractions. They use properties of simple continued fractions to obtain the following results which had previously been obtained by the reviewer [Bull. Amer. Math. Soc. **46**, 124–129 (1940); these Rev. **1**, 203] using other methods: (i) For any α there exist infinitely many p/q of each of the three classes satisfying (1) when $k\geqq 1$; (ii) If $k<1$, there exist irrationals α, everywhere dense, for which (1) holds for only a finite number of p/q of a given class. The distribution of approximants of a simple continued fraction among the three classes is discussed. The authors do not cite a paper of R. M. Robinson [Duke Math. J. **7**, 354–359 (1940); these Rev. **2**, 149], in which present methods are used to obtain (i) and a strengthened form of (ii).

W. T. Scott (Evanston, Ill.).

ANALYSIS

Weinberger, H. F. An inequality with alternating signs. Proc. Nat. Acad. Sci. U. S. A. **38**, 611–613 (1952).

Let $a_1 \geqq a_2 \geqq \cdots \geqq a_m \geqq 0$. The author proves the inequality

$$(1) \qquad \sum_{k=1}^{m}(-1)^{k-1}a_k{}^r \geqq \left[\sum_{k=1}^{m}(-1)^{k-1}a_k\right], \quad r \geqq 1.$$

R. Bellman has observed to the reviewer that

$$(2) \qquad \sum_{k=1}^{m}(-1)^{k-1}f(a_k) \geqq f\left[\sum_{k=1}^{m}(-1)^{k-1}a_k\right]$$

for any continuously differentiable convex function f whatever, provided only that $f(0) \leqq 0$. Clearly we may assume that $f'(x) > 0$ on $(0, a_1)$. The proof follows on observing that the right side of (2) is not greater than the area under $y = f'(x)$ between $x = 0$ and $x = A$, where $A = \sum_{k=1}^{m}(-1)^{k-1}a_k$, and that the left-hand side is the area of a number of non-overlapping strips bounded by the curve $y = f'(x)$ and intervals of the x-axis, the intervals having total length $= A$. (1) follows on taking $f(x) = x^r$. *J. M. Danskin.*

*Korovkin, P. P. Neravenstva. [Inequalities.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952. 56 pp. .95 rubles.

No. 5 in the series "Popular lectures on mathematics."

Nikol'skiĭ, S. M. Quadrature formulas. Izvestiya Akad. Nauk SSSR. Ser. Mat. **16**, 181–196 (1952). (Russian)

Let $0 \leqq x_0 < x_1 < \cdots < x_m \leqq 1$ be a system of points of the interval $[0, 1]$ and $p_0, p_1, \cdots, p_m$ a system of positive numbers (weights). With every function $f(x)$ continuous on $[0, 1]$ we associate the functional

$$(*) \qquad L(f) = L(0, 1; f) = \sum_{0}^{m} p_k f(x_k)$$

and treat $L(f)$ as an approximation to $\int_0^1 f(x)dx$:

$$\int_0^1 f(x)dx \sim L(f).$$

Let us also assume that this formula is exact for every polynomial of degree $\leqq r-1$. Let now $[\alpha, \beta]$ be any interval. The formula $L(\alpha, \beta; f) = \sum_0^m p'_k f(x'_k)$ will be called similar to (*) if the system of points $\alpha, x'_0, x'_1, \cdots, x'_m, \beta$ is geometrically similar to the system $0, x_0, x_1, \cdots, x_m, 1$, and if $p'_k = (\beta - \alpha)p_k$ for all k. Then, for any polynomial of degree $\leqq r-1$, $L(\alpha, \beta; f)$ equals $\int_\alpha^\beta f dx$. The usual method of an approximate computation of an integral $\int_a^b f(x)dx$ is as follows. One splits $[a, b]$ into n equal parts $[\xi_k, \xi_{k+1}]$ $(k = 0, 1, \cdots, n-1)$, to each of these intervals one applies a formula similar to (*), and by addition one obtains the more complicated formula

$$(**) \qquad \int_a^b f(x)dx \sim \sum_{0}^{n-1} L(\xi_k, \xi_{k+1}; f).$$

In the present note, the author discusses the degree of accuracy of this procedure. Let $KW^{(r)}(a, b)$ be the class of functions having on $[a, b]$ the derivative $f^{(r)}(x)$ satisfying the inequality $|f^{(r)}(x)| \leqq K$. Let also $K_r(t)$ denote the function vanishing for $t \leqq 0$ and equal to t^{r-1} for $t > 0$, and

$$F(t) = r^{-1}(1-t)^r - \sum_{0}^{m} p_k K_r(x-t), \quad c_r = \frac{1}{(r-1)!}\int_0^1 |F(t)|\,dt.$$

The author shows that (1) if $f \in KW^{(r)}(a, b)$, then the difference of both sides in (**) does not exceed $(b-a)^{r+1}c_r K n^{-r}$ numerically, and there exists a function f_* of the class considered for which this inequality becomes an equality. (2) If $f^{(r)}$ exists and has modulus of continuity $\omega(\delta)$, then the difference considered does not exceed $(b-a)^{r+1}c_r\omega((b-a)/n)n^{-r}$. The author also considers estimates for

$$\int_a^{b_k} f dx - \sum_{0}^{k-1} L(\xi_k, \xi_{k+1}; f),$$

uniform in x. The results are extended to functions of two variables. *A. Zygmund* (Chicago, Ill.).

Stöhr, Alfred. Über einen integralartigen Grenzübergang bei Kettenbrüchen. Math. Nachr. **6**, 103–107 (1951).

Let $p_{ij}(x)$, $i, j = 1, 2$, be properly Riemann integrable on $a \leqq x \leqq b$ and let E denote a subdivision,

$$a = x_0 < x_1 < \cdots < x_N = b,$$

and a collection of numbers, p_{ijn}, $n = 1, 2, \cdots, N$. If M is a constant for which $\sum_{ij}\int_a^b |p_{ij}(x)|\,dx < M$, then for any $\epsilon > 0$ there exists E such that

$$\sum_{ij}\sum_{1}^{N}|p_{ijn}| < M, \quad \left|\int_a^x p_{ij}(t)dt - \sum_{1}^{n} p_{ijr}\right| < \epsilon,$$

provided $x_n \leqq x < x_{n+1}$, $n = 0, 1, \cdots, N-1$; such an E is said to belong to ϵ. The symbol $y_E(a, b; y_0)$ denotes the linear fractional function of y_0 obtained by adding the $(2N+1)$th partial quotient $y_0/1$ to the terminating continued fraction $K_1^N\{(1+p_{11n})/p_{21n} + (1+p_{22n})/p_{12n}\}$. It is shown that: (i) If $E_1, E_2, \cdots$ belong to $\epsilon_1, \epsilon_2, \cdots$, respectively, and $\lim \epsilon_r = 0$, then $y(a, b; y_0) = \lim y_{E_r}(a, b; y_0)$, which is a linear fractional function of y_0, depends only on a, b, y_0, and the $p_{ij}(x)$. (ii) The function $y(a, x; y_0)$ (continued fraction integral) is continuous at the common points of continuity of the $p_{ij}(x)$, and for $a \leqq x \leqq b$ is the solution of the Riccati equation,

$$y' = -p_{21}(x)y^2 + [p_{11}(x) - p_{22}(x)]y - p_{12}(x),$$

for which $y(a) = y_0$. Some additional properties of continued fraction integrals are developed. Proofs depend on elementary properties of continued fractions and on a differential equations existence theorem of the author [Math. Nachr. **6**, 97–102 (1951); these Rev. **13**, 589]. *W. T. Scott.*

Kober, H. A remark on a monotone singular function. Proc. Amer. Math. Soc. **3**, 425–427 (1952).

The author's extension of the classical monotone continuous singular function of Lebesgue [Trans. Amer. Math. Soc. **67**, 433–450 (1949); these Rev. **11**, 336] is here further generalized. Let $\alpha > \beta > 1$ (β not necessarily an integer as previously). For $0 \leqq x < \infty$, set $\gamma(x) = \sum_{-\infty}^{n}\alpha^{-n}$ $[\beta^n x]$, where $[u]$ is the largest integer not exceeding u. Put

$$c_{\alpha,\beta} = \lim_{\epsilon \to 0+} 1/\gamma(1+\epsilon).$$

Set $t = g(x) = c_{\alpha,\beta}\gamma(x)$, $\Lambda(t) = x = g^{-1}(t)$, $0 \leqq t < \infty$. For β an integer, $\Lambda(t)$ reduces to the earlier monotone singular function $\omega(t)$. Many of the results on $\omega(t)$ hold for $\Lambda(t)$, often with simpler proofs. The function $g(x)$ is a strictly-increasing jump function, $\Lambda(t)$ is monotone and continuous,

$$g(0) = \Lambda(0) = 0, \quad g(1-) = \Lambda(1) = 1, \quad g(x) = \alpha g(x/\beta),$$
$$\Lambda(t) = \beta\Lambda(t/\alpha), \quad \Lambda(t+t') \leqq \Lambda(t) + \Lambda(t')$$

and $g(x) \to x$, $\Lambda(t) \to t$ for $\alpha - \beta \to 0$. There are indicated further applications to the approximation to a continuous function

of a single real variable by step functions, and to a continuous function of two real variables by nearly analytic functions. *E. N. Nilson* (Hartford, Conn.).

Harrik, I. Yu. On a problem of the constructive theory of functions connected with the investigation of convergence of variational processes. Doklady Akad. Nauk SSSR (N.S.) **80**, 25–28 (1951). (Russian)

The author presents, without proof, theorems of which the following is a sample: Let D be a closed 2-dimensional region bounded by the contour Γ whose equation is $\varphi(x, y)=0$. It is assumed that $\varphi(x, y)$ is defined in an open set containing D. Further, assume $\varphi(x, y)>0$ in D, and that $\varphi(x, y)$ is continuously differentiable in D and finally that $|\text{grad } \varphi(x, y)|>0$ on Γ. Then if $u(x, y)=0$ on Γ and $u(x, y)$ is continuous in D, one can find a sequence of polynomials $P_n(x, y)$, $P_n(x, y)$ of degree not more than n in each variable, such that $\|u-\varphi P_n\|_C \leqq A\omega_C(u; 1/n)$. Here $\|\cdots\|_C$ is the Banach norm for continuous functions, and

$$\omega_E(u, t)=\sup_{[\alpha^2+\beta^2]^{1/2}\leqq t}\|u^{\alpha,\beta}-u\|_E,$$

for any Banach space E, where

$$u^{\alpha,\beta}=\begin{cases} u(x+\alpha, y+\beta) & \text{for } (x+\alpha, y+\beta) \text{ in } D \\ 0 & \text{otherwise.}\end{cases}$$

A is a constant. The other theorems given concern themselves with convergence in other norms and with convergence of derivatives. *B. R. Gelbaum.*

Il'in, V. P. On the convergence of variational processes. Doklady Akad. Nauk SSSR (N.S.) **81**, 137–140 (1951). (Russian)

The author gives, without proof, theorems which guarantee the uniform convergence of minimizing sequences such as are used in problems of the calculus of variations, with side conditions. The assumption that a solution to the variational problem exists is always made. A sample theorem is: Let the differential equation considered be:

$$-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right)+b(x_1, \cdots, x_n)u=f(x_1, \cdots, x_n) \quad (a_{ij}=a_{ji}).$$

Let the side conditions be one of the following three types: (1) $u=0$ on Γ (the boundary of the region D for which the problem is posed); (2) $\sum_{j=1}^{n}\cos(n, x_j)\sum_{i=1}^{n}a_{ij}\partial u/\partial x_i=0$ on Γ (where n is the inward normal on Γ); or (3)

$$\sum_{j=1}^{n}\cos(n, x_j)\sum_{i=1}^{n}a_{ij}\frac{\partial u}{\partial x_i}+\sigma u=0 \text{ on } \Gamma \quad (\sigma(P)\geqq\sigma_0>0).$$

The a_{ij} are assumed continuous and continuously differentiable in $D\cup\Gamma$; b is assumed non-negative, bounded and measurable in D, and f is assumed square-summable in D. For boundary conditions (1) and (2) let

$$F(u)=\int\cdots\int_{D}\int\left(\sum_{i,j=1}^{n}a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}+bu^2-2fu\right)dx_1\cdots dx_n.$$

For boundary condition (3) let $F(u)$ be as above with the term $\int\int\cdots\int_{\Gamma}\sigma u^2 d\Gamma$ added. Assume $\int_{D_1}(\partial u/\partial r)^2 dr\leqq L_D^2$ where r is an arbitrary direction and D_1 is an arbitrary section made by a line in the direction r. Let u_k be a sequence of functions satisfying the same boundary conditions, continuous and continuously differentiable and such that $\int_{D_1}(\partial u_k/\partial r)^2 dr\leqq L_k^2$. Let $F(u_k)-F(u)=A_k^2\to 0$, $A_kL_k^{n-2}\to 0$ for $n>2$, $A_kL_k^{2\alpha}\to 0$ for $n=2$, α, a small positive number (for conditions (2) and (3)) and $A_k \ln L_k=0$ for $n=2$ and condition (1). Then the u_k are uniformly convergent to u in $D\cup\Gamma$ and $|u-u_k|\leqq C_1[A_k(L_0^2+L_k^2)^{\frac{1}{2}(n-2)}]^{1/(n-1)}$ for $n>2$; $|u-u_k|\leqq C_2[A_k(L_0^2+L_k^2)]^{1/(1+2\alpha)}+C_3A_k$ for $n=2$ and conditions (2) or (3) and

$$|u-u_k|\leqq C_4A_k[\ln(d(L_0^2+L_k^2)/A_k^2)]^{1/2}+C_5A_k$$

for $n=2$ and condition (1). C_i are constants, d is the diameter of D and Γ is assumed piece-wise smooth.

B. R. Gelbaum (Minneapolis, Minn.).

Harrik, I. Yu. On a problem of approximation of functions connected with investigation of the convergence of variational processes. Doklady Akad. Nauk SSSR (N.S.) **81**, 157–160 (1951). (Russian)

Continuing in the vein of the first of the two papers reviewed above, the author gives, without proof, more theorems on convergence of approximation methods. In particular the following is quoted: Let D, Γ be as in the preceding reviews. Let φ be defined in an open set containing D, $\varphi=0$ on Γ, $\varphi\neq 0$ off Γ, φ continuously differentiable $k\geqq 1$ times, $\partial^k\varphi/\partial x^r\partial y^{k-r}\in \text{Lip } 1$ for $0\leqq r\leqq k$, grad $\varphi\neq 0$ on Γ. Let u be continuously differentiable in D and $u=0$ on Γ. Then there is a sequence of polynomials $P_n(x, y)$, P_n of degree n or less in x and in y, such that

$$\|u-\varphi P_n\|_C=O\left(\frac{\omega_C^{[k]}(u; 1/n)}{n^k}\right)$$

where $\omega_C^{[k]}(u; 1/n)=\max_{0\leqq r\leqq k}\omega_C(\partial^k u/\partial x^r\partial y^{k-r}; 1/n)$ and $\omega_C(u, \delta)$ is the modulus of continuity of u. Furthermore, for $k\geqq 2$ one has

$$\left\|\frac{\partial^r}{\partial x^r\partial y^{r-r}}(u-\varphi P_n)\right\|_C=O\left(\frac{\omega_C^{[k]}(u; 1/n)}{n^{k-r}}\right)$$

for $0\leqq r\leqq r<k$. This and allied results are declared to have applicability in the (Rayleigh)-Ritz method of the calculus of variations. *B. R. Gelbaum* (Minneapolis, Minn.).

Theory of Sets, Theory of Functions of Real Variables

Sierpiński, W. Sur les opérations dans l'ensemble à 3 éléments. Ann. Soc. Polon. Math. **24** (1951), 13–18 (1952).

Of the 3^3 possible binary operations on a set of 3 elements the 12 which admit both right and left inverse are described.

W. Gustin (Bloomington, Ind.).

Sierpiński, W. Un théorème sur les familles de fonctions et son application aux espaces topologiques (solution d'un problème de R. Sikorski). Colloquium Math. **2** (1951), 198–201 (1952).

The author proves the following theorem. Let E be a set, F a family of real-valued functions $f(x)$ defined on E which satisfies the following properties: (1) The set of $x\in E$ for which $f(x)\neq 0$ has power $\leqq m$ (where m is an infinite cardinal number). (2) If f and g are two functions of the family F, there always exists an $x\in E$ so that $f(x)\neq g(x)$ and $f(x)\neq 0$, $g(x)\neq 0$. The conclusion then is that the power of the family F is $\leqq 2^m$. The author then uses this theorem to prove a conjecture of Sikorski. *P. Erdös.*

Sierpiński, W. Sur une propriété paradoxale de l'espace à trois dimensions équivalente à l'hypothèse du continu. Rend. Circ. Mat. Palermo (2) **1**, 7–10 (1952).

The author proves that the continuum hypothesis can be expressed equivalently as follows in terms of elementary geometry without using the notion of infinity: Euclidean three-dimensional space E is the union of three sets, E_1, E_2, E_3, such that, if OX_1, OX_2, OX_3 denote the three coordinate axes of the space E, every straight line parallel to the OX_i-axis contains only a finite number of points of E_i $(i=1, 2, 3)$.

F. Bagemihl (Rochester, N. Y.).

Sierpiński, Wacław. Sur quelques propositions concernant la puissance du continu. Fund. Math. **38**, 1–13 (1951).

Kuratowski, Casimir. Sur une caractérisation des alephs. Fund. Math. **38**, 14–17 (1951).

Sikorski, Roman. A characterization of alephs. Fund. Math. **38**, 18–22 (1951).

Sierpiński's paper contains, in addition to a generalization of the theorem stated in the preceding review, propositions of a similar geometrical nature, some of which are shown to be equivalent to the continuum hypothesis or to the hypothesis that $2^{\aleph_0} \leqq \aleph_1$. Denote by X^n the Cartesian product of n replicas of a set X. A subset A of X^n is said to be of power less than $\mathfrak{m}$ in the direction of the kth axis, if, for every element $(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n)$ of X^{n-1}, there are fewer than $\mathfrak{m}$ elements x_k of X such that

$$(x_1, \cdots, x_{k-1}, x_k, x_{k+1}, \cdots, x_n) \in A.$$

Kuratowski, generalizing certain of Sierpiński's results, proves the following: The inequality $|X| < \aleph_{n+n}$ is equivalent to the existence of a system of $n+1$ sets, $A_1, \cdots, A_{n+1}$, such that $X^{n+1} = A_1 + \cdots + A_{n+1}$, and, for $k \leqq n+1$, the set A_k is of power less than $\aleph_n$ in the direction of the kth axis. This enables him to define the alephs of finite index without the intervention of transfinite cardinal or ordinal numbers: In order that a set X be of power $\aleph_n$ (n a nonnegative integer), it is necessary and sufficient that the set X^{n+2} admit of a decomposition into $n+2$ sets $A_1, \cdots, A_{n+2}$ such that, for $k \leqq n+2$, A_k is finite in the direction of the kth axis, and the set X^{n+1} not admit of a decomposition of this sort into $n+1$ sets. Sikorski generalizes theorems of Sierpiński and Kuratowski as follows: Denote the class of all subsets of i elements of $\{1, 2, \cdots, j\}$ $(i \leqq j)$ by $I_{i,j}$. Let $\Lambda \in I_{i,j}$. Every subset P of X^j of the form $P = Y_1 \times Y_2 \times \cdots \times Y_j$, where $Y_\nu = \{a_\nu\}$ $(a_\nu \in X)$ if $\nu \in \Lambda$, and $Y_\nu = X$ for ν non-ε Λ, is called a Λ-set. Let n and k be positive integers, and let X be a nonempty set. In order that $|X| < \aleph_{n+n}$, it is necessary and sufficient that X^{n+k} be the union of $\binom{n+k}{n}$ sets E_Λ $(\Lambda \in I_{n,n+k})$ such that $|PE_\Lambda| < \aleph_n$ for every Λ-set P. Taking X to be the set of real numbers, the cases $\alpha=0$, $n=2$ and $\alpha=1$, $n=1$ yield propositions about Euclidean space which are equivalent to the continuum hypothesis. *F. Bagemihl.*

Sur une famille singulière d'ensembles de nombres naturels. Travail collectif, rédigé par S. Hartman. Colloquium Math. **2** (1951), 245–248 (1952).

The authors, answering a question of Hartman, construct $2^{\aleph_0}$ sequences of integers each having density 1 so that the intersection of any nondenumerable family of these sequences is finite. Several constructions are given; each of them depend on the continuum hypothesis. *P. Erdős.*

Sierpiński, Wacław. Un théoreme sur les fonctions d'ensemble. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. **42** (1949), 18–22 (1952). (French. Polish summary)

Let S be a set, and let $E_1, \cdots, E_n$ be subsets whose union is S. For $1 \leqq i_1 < i_2 < \cdots < i_k \leqq n$ consider the $2^n - 1$ products $E_{i_1} E_{i_2} \cdots E_{i_k}$ (which may not be all distinct). Let $N_1, N_2, \cdots$ be those products for which k is odd, and let $P_1, P_2, \cdots$ be those for which k is even. Let F be a set-function defined on the subsets of S with values in a set M with an associative and commutative operation $+$, such that

$$F(A+B) = F(A) + F(B)$$

when AB is void. It is proved that

$$F(S) + F(P_1) + F(P_2) + \cdots = F(N_1) + F(N_2) + \cdots.$$

It is observed that the case in which $F(A)$ is the cardinal number of A has been used in combinatorial analysis.

R. Arens (Los Angeles, Calif.).

Denjoy, Arnaud. Addition conventionnelle et figurée des permutations. C. R. Acad. Sci. Paris **234**, 770–773 (1952).

Denjoy, Arnaud. Les suites finies d'entiers positifs génératrices de permutations. C. R. Acad. Sci. Paris **234**, 906–908 (1952).

Denjoy, Arnaud. La définition d'un nombre ordinal non préconçu. C. R. Acad. Sci. Paris **234**, 1001–1004 (1952).

Denjoy, Arnaud. La figuration arithmétique des permutations clivées. C. R. Acad. Sci. Paris **234**, 1102–1105 (1952).

Denjoy, Arnaud. Les suites canoniques des nombres de seconde espèce de la classe II. C. R. Acad. Sci. Paris **234**, 2033–2037 (1952).

Denjoy, Arnaud. Les suites canoniques. C. R. Acad. Sci. Paris **234**, 2129–2131 (1952).

Denjoy Arnaud. Les suites canoniques. C. R. Acad. Sci. Paris **234**, 2405–2407 (1952).

Nelle sette note suindicate, l'autore riassume il libro II ("L'arithmétisation du transfini", in corso di stampa) della sua opera dal titolo: "L'énumération transfinie". Dati il carattere riassuntivo delle note, l'originalità e la profondità delle ricerche contenute nel detto libro II, la stretta dipendenza dal libro I [Gauthier-Villars, Paris, 1946; questi Rev. **8**, 254], la personalità, la novità, la ricchezza sia dei concetti che della terminologia, si ritiene che l'analisi di queste note debba farsi opportunamente rientrare in quella che verrà data, in propria sede e con tutta l'ampiezza necessaria, del libro II stesso. Perciò ci si limita qui ai seguenti cenni brevissimi.

Le permutazioni, cui s'accenna nei titoli, sono tutti gli ordinamenti (lineari aperti) che possono attribuirsi alla successione dei numeri naturali. Vengono particolarmente studiate le permutazioni di certi tipi speciali, e le cosidette "somme di successioni, semplicemente infinite, di permutazioni". Siffatte somme vengono utilizzate per definire, sia costruttivamente, sia direttamente, i numeri transfiniti della seconda classe. Viene risolto l'importante problema (posto da N. Lusin ma da questi ritenuto insolubile) di definire un tipo di ben ordinamento, il cui rango (transfinito) sia praticamente impossibile a prevedersi, e neppure a scoprirsi. Viene proseguito, da vari punti di vista, lo studio del cosiddetto "sviluppo neperiano" d'un numero reale, e del modo con cui un tale sviluppo può considerarsi rappresentativo d'una permutazione.

Sono poi profondamente studiate certe particolari successioni $\{\varphi(\gamma)\}$ $[\gamma=1, 2, \cdots, \omega, \cdots; \gamma<\Omega; \omega\leqq\varphi(\gamma)<\Omega]$, ben ordinate, di numeri $\varphi(\gamma)$ ordinali transfiniti di seconda classe: successioni dette regolari, in quanto crescenti e topologicamente chiuse [cioè tali che $\varphi(\gamma)=\lim_{\gamma'<\gamma}\varphi(\gamma')$, per ogni γ di seconda specie]. Casi particolarmente semplici delle successioni regolari, sono le successioni dette canoniche, utilizzate per determinare i numeri di seconda specie (ciascuna successione canonica tende al numero che rappresenta). Questo studio è fatto partendo da opportuni, semplici postulati. *T. Viola* (Roma).

Vol'pert, A. Ya. On homeomorphism of denumerable sets. Mat. Sbornik N.S. **29(71)**, 677–698 (1951). (Russian)

The following theorems are proved. I. A countably infinite set A of (real) numbers is homeomorphic to a set of numbers ordinally similar to the set R of rational numbers if and only if A contains no isolated point of accumulation, i.e., $A\cap(A'-A'')=0$. II. The set of all countably infinite, everywhere dense order types has cardinal number $\exp(\aleph_0)$. III. For a given set B ordinally similar to R, there are $\exp(\aleph_0)$ distinct order types of sets homeomorphic to B. *E. Hewitt* (Seattle, Wash.).

Inagaki, Takeshi. Sur deux théorèmes concernant un ensemble partiellement ordonné. Math. J. Okayama Univ. **1**, 167–176 (1952).

L'A. s'occupe de questions liées aux recherches de Dedekind [Was sind und was sollen die Zahlen?, Gesammelte Math. Werke, Bd. III, Vieweg, Braunschweig, 1932, pp. 335–391], de Zermelo, et de Zorn. Il prouve ceci (Théorème 1): Soit E un ensemble ordonné inductif au sens faible; s'il existe une transformation univoque f de E en lui-même telle que $x\leqq f(x)$ $(x\varepsilon E)$, alors il existe une transformation K de E en lui-même vérifiant les conditions $\alpha)-\gamma)$ que voici: $\alpha)$ Quel que soit $a\varepsilon E$, $K(a)$ est une chaîne bien ordonnée contenant a; $\beta)$ $f(K(a))\subseteq K(a)$; $\gamma)$ quelle que soit la chaîne $A\subseteq K(a)$, alors $\sup A$ est vide ou un élément de $K(a)$. On en déduit l'équivalence des trois propositions (1)–(3) que voici: (1) L'axiome de choix; (2) lemme de Zorn; (3) dans chaque ensemble ordonné non vide inductif au sens faible, E, il existe une transformation K de E en lui-même telle que: $\alpha)$ pour tout $a\varepsilon E$, $K(a)$ est une chaîne bien ordonnée contenant a; $\beta)$ si $A\subseteq K(a)$ et si $\sup A$ existe, alors $\sup A\varepsilon K(a)$; $\gamma)$ un point de E n'appartenant pas à $K(a)$ ne peut pas majorer $K(a)$. (Inductivité faible de E signifie l'existence pour chaque chaîne $C\subseteq E$ d'une plus petite borne supérieure (supremum), $\sup C$, pourvu qu'il existe un $x\varepsilon E$ majorant C.) Il est intéressant de comparer la démonstration du th. 1 à celle de $F\to M$ de Witt [Math. Nachr. **4**, 434–438 (1951); ces Rev. **12**, 596]. *Đ. Kurepa* (Zagreb).

Katětov, M. Measures in fully normal spaces. Fund. Math. **38**, 73–84 (1951).

Let P be a completely regular space. Let $\mathbf{F}(P)$ and $\mathbf{F}^*(P)$ be, respectively, the families of closed sets and sets of the form $f^{-1}(0)$ for continuous real-valued f on P. Let $\mathbf{B}(P)$ and $\mathbf{B}^*(P)$ be the σ-algebras generated by $\mathbf{F}(P)$ and $\mathbf{F}^*(P)$, respectively; these σ-algebras are the Borel and Baire sets of P, respectively. A σ-additive, non-negative, real-valued, finite set function μ defined on a σ-algebra $\mathbf{B}\supset\mathbf{B}^*(P)$ is a measure on P, under the author's terminology; if $\mathbf{B}=\mathbf{B}(P)$ $(\mathbf{B}^*(P))$, then μ is a Borel (Baire) measure. A closed set $Q\subset P$ semi-reduces a measure if $\mu(G)>0$ for all open $G\varepsilon\mathbf{B}$ such that $G\cap Q\neq 0$ and $\mu(F)=0$ for all closed $F\varepsilon\mathbf{B}$ such that $F\cap Q=0$. If, in addition, $Q\varepsilon\mathbf{B}$, then Q is said to reduce μ. The author goes through certain topological preliminaries, of interest for their own sake, and then makes the following contributions to the difficult problem of reducibility. Let P be paracompact. Then every [2-valued] Borel and Baire measure on P is semi-reducible if and only if every [2-valued] Borel measure on every closed discrete subspace of P is reducible. If P is hereditarily paracompact, then every [2-valued] Borel measure in P is reducible if and only if the same is true for every discrete subspace of P. Turning next to the reviewer's Q-spaces [Trans. Amer. Math. Soc. **64**, 45–99 (1948); these Rev. **10**, 126], the author shows that a paracompact space P is a Q-space if and only if every closed discrete subspace of P is a Q-space. A metric space P is a Q-space if every discrete subspace of P is a Q-space. This extends a result of the reviewer [loc. cit., Theorem 53]. *E. Hewitt* (Seattle, Wash.).

Enomoto, Shizu. A lattice-theoretic treatment of measures and integrals. Proc. Japan Acad. **28**, 14–18 (1952).

The author relaxes the last axiom in Carathéodory's definition of a set of somas [S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1938**, 27–69]. Then, after a review of several propositions due to Carathéorory and corresponding modification of some of them, he states (without proof) a theorem corresponding to the extension theorem of Kolmogoroff-Hopf. *H. M. Schaerf* (St. Louis, Mo.).

Łoś, J. Un problème concernant le prolongement des fonctions aux σ-mesures. Colloquium Math. **2** (1951) 271–274 (1952).

If X is a subset of a Boolean algebra B, let X' denote the Boolean subalgebra generated by X. A real-valued function f defined on X satisfies condition C_τ if for every countable subset Y of B there exists a countably additive normalized measure μ on Y' such that $\mu(a)=f(a)$ for every a in $X\cap Y'$. The author shows that C_τ does not imply that f can be extended to a countably additive normalized measure on B and he raises the problem of whether or not the implication becomes valid under the additional assumption $X'=B$. A suitable (somewhat complicated) strengthening of C_τ is shown to be sufficient for the solution of the extension problem. *P. R. Halmos* (Chicago, Ill.).

Kawada, Yukiyosi. A note on integration. Sci. Papers Coll. Gen. Ed. Univ. Tokyo **1**, 15–18 (1951).

The author shows that most of the familiar facts about outer measures hold for any extended real-valued functional L on the class $\mathfrak{F}$ of all non-negative functions on a set X provided that (a) L has the following properties of an outer measure (considered as a functional on the class of all characteristic functions of subsets of X): it is non-negative, monotone, countably subadditive, and vanishes at 0; (b) an element φ of $\mathfrak{F}$ is called measurable if $\psi\varepsilon\mathfrak{F}$ implies

$$L\psi\geqq L(\psi\cap\varphi)+L[(\psi-\varphi)\cap\varphi].$$

As an application, he proves an extension theorem for functionals which generalizes the criterion for extensibility of a finitely additive measure to a countably additive one and was proved in similar form by other authors. *H. M. Schaerf* (St. Louis, Mo.).

Alexiewicz, Andrzej. On Denjoy integrals of abstract functions. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. **41**(1948), 97–129 (1950). (English. Polish summary)

The author considers functions defined on an interval of the real axis to a Banach space. He defines a Denjoy-

Bochner integral for a suitable subset of this class of functions. The relationship of this integral to the Lebesgue-Bochner integral [Fund. Math. **20**, 262–276 (1933)] is analogous to the relation of the classical Denjoy integral to the Lebesgue integral. Also he defines a Denjoy-Pettis integral where the role of the "strong" integral of Bochner is taken by the "weak" integral of Pettis [Trans. Amer. Math. Soc. **44**, 277–304 (1938)]. Each of these integrals is given an equivalent constructive and descriptive definition. For the functions involved, relations between the Denjoy integration and appropriate notions of differentiation are determined and properties of the integral obtained. The definitions are too lengthy to be reproduced here.

B. Yood (Ithaca, N. Y.).

Džvaršeišvili, A. G. On a double integral of Denjoy-Čelidze. Soobščeniya Akad. Nauk Gruzin. SSR. **12**, 193–199 (1951). (Russian)

If $R_0=[(a, b)(c, d)]$ is a 2-dimensional interval, the function $f(x, y)$ is said to be integrable on R_0 in the sense of Denjoy-Čelidze (D-T) if, almost everywhere on R_0, $f(x, y)=D_{xy}F(x, y)$ holds, where $F(x, y)$ is a generalized absolutely continuous function on R_0 in the sense of Čelidze [Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **15** (1947) (unavailable); **2**, 109–142 (1937)]. The author proves some properties of this integral, for instance: Theorem 3: Let $f(x, y)$ be (D-T)-integrable on R_0 and $\Phi(R)=\int\int_R f(x, y)dxdy$. If $\Phi(R)$ is absolutely continuous (in the sense of Čelidze) on the set $E=[P(c, d)]+[Q(a, b)]$, where P and Q are closed subsets of (a, b) and (c, d) respectively, then $f(x, y)$ is summable (Lebesgue) on E.

M. Cotlar (Chicago, Ill.).

Kirsch, A. Über Zerlegungsgleichheit von Funktionen und Integration in abstrakten Räumen. Math. Ann. **124**, 343–363 (1952).

The author studies the problem of existence of linear functionals on various classes of functions subject to various requirements of invariance under a group acting on the underlying set. Roughly speaking, the circle of ideas treated in this technically elegant paper bears the same relation to the concepts associated with the names of Banach and Tarski as does the concept of a real-valued function to that of a set.

P. R. Halmos (Chicago, Ill.).

Grinblyum, M. M. General definition of an operator integral. Doklady Akad. Nauk SSSR (N.S.) **84**, 661–664 (1952). (Russian)

Continuing a sequence of notes [same Doklady **70**, 941–944; **71** (1950), 5–8; **81**, 345–348 (1951); these Rev. **11**, 525, 601; **13**, 470] this note discusses a definition of a Lebesgue integral with respect to a spectral measure $P(\lambda)$, $-\infty<\lambda<\infty$, whose values are projections in a reflexive Banach space [see Lorch, Trans. Amer. Math. Soc. **45**, 217–234 (1939)]. A first step toward a corresponding generalization of Denjoy totalization is described.

M. M. Day (Urbana, Ill.).

Pi Calleja, Pedro. On the concept of the integral. Revista Soc. Cubana Ci. Fís. Mat. **2**, 188–199 (1952). (Spanish)

Continued from an earlier set of expository papers in the same journal **1**, 19–27, 58–62 (1942); 88–91, 123–127 (1943); 164–169 (1944); these Rev. **6**, 44.

Besicovitch, A. S. On existence of subsets of finite measure of sets of infinite measure. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 339–344 (1952).

It is proved that any $F_{\sigma\delta}$ set of infinite Hausdorff Λ^s-measure contains (closed) subsets of all finite Λ^s-measures. Whether $F_{\sigma\delta}$ can be replaced by Borel set remains undecided.

H. D. Ursell (Leeds).

Gomes, Ruy Luís. Example of a set not Lebesgue measurable. Gaz. Mat., Lisboa **13**, no. 51, 4–6 (1952). (Portuguese)

Viola, Tullio. Etude des propriétés géométriques de certains domaines d'intégration, qu'on rencontre dans quelques problèmes de physique mathématique. C. R. Acad. Sci. Paris **234**, 2585–2587 (1952).

Remarks on the superficial measure of parts of the boundary of certain domains [cf. Viola, same C. R. **234**, 2513–2515 (1952); these Rev. **13**, 925].

L. M. Graves.

Hadwiger, Hugo. Ueber addierbare Intervallfunktionale. Tôhoku Math. J. (2) **4**, 33–37 (1952).

The author characterizes all finitely additive and translation-invariant interval-functions in k-dimensional space in terms of Hamel bases (Theorem 1) and proves that two intervals are equivalent by finite decomposition if and only if every such interval-function assumes equal values on them (Theorem 2). From these facts he is able to give an elegant deduction of the known rationality conditions for the equivalence of two intervals (Corollary 1) and for the decomposability of one interval into cubes (Corollary 2).

P. R. Halmos (Chicago, Ill.).

Bobynin, M. N. On a theorem of the theory of completely additive set functions. Uspehi Matem. Nauk (N.S.) **7**, no. 3(49), 113–120 (1952). (Russian)

This (posthumous) note contains results selected from an unpublished manuscript of the author, with some supplementary remarks by M. G. Kreĭn. Theorem. Suppose that Φ is a family of completely additive set functions defined on a Borel field T of subsets of a set R, and that $R\in T$. Suppose also that for each set E of the field T the set of numbers $\{f(E)\,|\,f\in\Phi\}$ is bounded. Then the set of numbers $\{f(E)\,|\,f\in\Phi, E\in T\}$ is bounded.

M. M. Day.

Hayes, Charles A., Jr. Differentiation with respect to ϕ-pseudo-strong blankets and related problems. Proc. Amer. Math. Soc. **3**, 283–296 (1952).

A chaque point x d'un espace métrique, supposons associée une famille d'ensembles boréliens bornés $F(x)$ de sorte que toute boule de centre x contienne de ces ensembles [$F(x)$ est appelé "blanket" par l'auteur]. On considère des "mesures" dénombrablement additives, définies sur les boréliens, non négatives, et finies pour tout ensemble borné. Soit Φ une mesure fixe, $G(x)$ un "blanket" quelconque extrait de $F(x)$, A un ensemble quelconque, x_i une suite dénombrable de points de A, $A_i\in G(x_i)$ une suite d'ensemble, B leur réunion, $C=A-A\cap B$. A. P. Morse a démontré le théorème suivant [Trans. Amer. Math. Soc. **55**, 205–235 (1944); ces Rev. **5**, 231]: Quels que soient $G(x)$ et A, supposons que les A_i puissent être choisis disjoints et tels que $\Phi(C)=0$. Alors $F(x)$ est dit "Φ-strong-blanket" et toute "mesure" Ψ a, par rapport à Φ, une dérivée évaluée avec $F(x)$ qui existe et est finie en tout point sauf sur un ensemble pour lequel Φ est nulle. Il en résulte que les théorèmes de Lebesgue sont applicables.

Le but essentiel du mémoire est de démontrer que le résultat subsiste en assujetissant $F(x)$ aux conditions moins restrictives suivantes: quels que soient $G(x)$, A, $\epsilon>0$, et la mesure X, on peut choisir les A_i de sorte que: (1) $\overline{\Phi}(C)<\epsilon$, et que (2) $\int_B[\sum\phi_{A_i}(x)-1]dX<\epsilon$ ($\overline{\Phi}$ désignant la mesure extérieure associée à Φ, et $\phi_E(x)$ étant la fonction caractéristique de E). $F(x)$ est alors dit "Φ-pseudo-strong-blanket". L'auteur donne un exemple montrant que la condition (2), supposée verifiée pour $X=\Phi$ seulement, jointe à la condition (1), est insuffisante.

Une condition analogue permettant d'affirmer l'exactitude des théorèmes de Lebesgue a été donnée par le rapporteur [C. R. Acad. Sci. Paris **224**, 1197–1198 (1947); ces Rev. **8**, 572]. Il s'agissait d'un cas plus général, l'espace n'étant pas supposé métrique, et $F(x)$ étant remplacé par un filtre $\mathfrak{F}_x$ associé à chaque point x, sur l'ensemble des fonctions mesurables non négatives. *R. de Possel* (Alger).

Hayes, Charles A. Differentiation of some classes of set functions. Proc. Cambridge Philos. Soc. **48**, 374–382 (1952).

Let $\mathcal{S}$ be a metric space and $\phi(\beta)$ a Carathéodory outer measure defined for every set $\beta\subset\mathcal{S}$. F is a blanket with domain $A\subset\mathcal{S}$ if, for each $x\in A$, $F(x)$ is a family of subsets of $\mathcal{S}$ such that (i) if $\beta\in F(x)$, diam $\beta<\infty$, and (ii) $\inf_{\beta\in F(x)}$ diam $(\beta\cup\hat{x})=0$. [$\hat{x}$ denotes the set with the single member x.] The author defines a class S_p ($p>1$) of blankets so that $F\in S_p$ if and only if F and all its subblankets have the following properties: (i) each member of $F(A)$ is a Borel set; (ii) given $\epsilon>0$ there exists a countable subfamily $\mathfrak{G}\subset F(A)$ such that (a) $\phi(A-\sigma\mathfrak{G})=0$ and (b) $\int_{\sigma\mathfrak{G}}\{P_{\mathfrak{G}}(x)-1\}^p d\phi<\epsilon$, where $\sigma\mathfrak{G}$ denotes the union of the sets in $\mathfrak{G}$, and $P_{\mathfrak{G}}(x)$ is the number of members of $\mathfrak{G}$ to which x belongs. Condition (b) thus relaxes somewhat the ordinary Vitali condition that the sets in $\mathfrak{G}$ shall be non-overlapping. It is shown that blankets of the class S_p are suitable for the differentiation of certain set functions. If $\psi(\beta)$ is any set function then $D(\psi,\phi,F,x)$ is the limit (if it exists) of $\psi(\beta)/\phi(\beta)$ as diam $(\beta\cup\hat{x})\to 0$ with $\beta\in F(x)$. The author proves that $D(\psi,\phi,F,x)$ exists and is finite if $F\in S_p$ and $\psi(\beta)=\int_\beta u d\phi$ where $\int_{\mathcal{S}}u^{q'}d\phi<\infty$, provided that $q'\geqq p/(p-1)$. He also shows by an example that this property may fail if $q'<p/(p-1)$. *U. S. Haslam-Jones* (Oxford).

Pukánszky, L., and Rényi, A. On the approximation of measurable functions. Publ. Math. Debrecen **2**, 146–149 (1951).

Sia $\{f_n(x)\}$ $(n=1,2,\cdots)$ una successione di funzioni misurabili nell'intervallo $[0,1]$, tali che $0\leqq f_n(x)\leqq 1$ per ogni $n=1,2,\cdots$. La successione sia massimale, cioè, per ogni coppia di valori distinti x_1, x_2 di x in $[0,1]$, non appartenenti, nè l'uno nè l'altro, ad un certo insieme di misura nulla, esista almeno una funzione $f_n(x)$ tale che $f_n(x_1)\neq f_n(x_2)$. Sia infine $f(x)$ una qualunque funzione misurabile in $[0,1]$, tale che risulti $a\leqq f(x)\leqq b$ in quasi tutto $[0,1]$ (a, b costanti). Si dimostra allora che, per ogni coppia di numeri positivi ϵ, δ e per un intero $m>0$ sufficientemente elevato, è possibile determinare un polinomio $P_m[f_1(x),f_2(x),\cdots,f_m(x)]$ delle funzioni $f_1(x),f_2(x),\cdots,f_m(x)$, e un insieme $E\subset[0,1]$, in modo che risulti:

$$\text{mis } E>1-\delta,\ |f(x)-P_m[f_1(x),f_2(x),\cdots,f_m(x)]|<\epsilon \text{ per } x\in E,$$

e inoltre

$$a\leqq P_m[f_1(x),f_2(x),\cdots,f_m(x)]\leqq b \text{ per } x\in[0,1].$$

Gli autori deducono alcune interessanti conseguenze da questa proposizione e concludono con osservazioni critiche. *T. Viola* (Roma).

Fast, H. Sur la convergence statistique. Colloquium Math. **2** (1951), 241–244 (1952).

Steinhaus introduced the following definitions: A sequence of functions $f_n(x)$ is said to converge statistically to $f(x)$ if all the functions $f_n(x)$ are measurable (L) and if for almost all x_0 there exists a sequence n_i of density 1 (the sequence may depend on x_0) so that $\lim f_{n_i}(x_0)=f(x_0)$. Denote by $m_n(\epsilon)$ the measure of the set in x for which $|f_n(x)-f(x)|\geqq\epsilon$. A sequence of functions $f_n(x)$ is said to converge asymptotically statistically to $f(x)$ if to every $\epsilon>0$ there exists a sequence n_i of density 1 so that $\lim m_{n_i}(\epsilon)=0$. Steinhaus proved that for sequences of measurable functions $f_n(x)$ the two definitions are equivalent. The author gives a simpler proof for this result. *P. Erdős.*

Pipes, C. J. Generalizations of a theorem of Sierpinski and Zygmund on continuous functions. Proc. Amer. Math. Soc. **3**, 237–243 (1952).

Una funzione reale $f(x)$, definita in un certo insieme non numerabile S di valori reali x, è detta d-continua in un certo punto x di S e rispetto ad S, se ad ogni numero $\epsilon>0$ è possibile far corrispondere un intorno $I=I(x,\epsilon)$ di x e un insieme numerabile $D=D(x,\epsilon)$ in modo che risulti $|f(x)-f(y)|<\epsilon$ per $y\in IS-D$. Nell'ordine d'idee dei due autori citati nel titolo e in forza di quanto da loro dimostrato [cfr. Fund. Math. **4**, 316–318 (1923)], viene costruita, sull'asse reale, una certa funzione $h(x)$ tale che, qualunque sia S, $h(x)$ non è d-continua, rispetto ad S, in ogni punto di S. La $h(x)$ può considerarsi un caso particolare di un tipo di funzioni che godono di proprietà molto generali: altri casi particolari notevoli vengono studiati. *T. Viola.*

Greco, Donato. Criteri di compattezza per insiemi di funzioni in n variabili indipendenti. Ricerche Mat. **1**, 124–144 (1952).

A function $u(x_1,\cdots,x_n)$ is said to be p-regular in a domain T if its partial derivatives of orders $\leqq(p-1)$ are continuous on T and its pth partial derivatives are continuous on $T-FT$. Suppose that the pth partial derivatives are summable with an exponent $m\geqq 1$ on T. Let Δ be any domain included in T, and write

$$J_{m,p}[u,\Delta]=\left[\int_\Delta\left\{\sum_{j_1,j_2,\cdots,j_p}^{1,\cdots,n}\left|\frac{\partial^p u}{\partial x_{j_1}\partial x_{j_2}\cdots\partial x_{j_p}}\right|^m\right\}d\tau\right]^{1/m}.$$

The frontier FT of the domain T is assumed to be p-regular, i.e., it consists of a finite number of regular varieties on each one of which the coordinate of the running point may be expressed as a function of class $C^{(p)}$ of the $n-1$ local parameters. The author proves that if the members of a family U of functions u of the type described above are equibounded in T, and if there is an $H>0$ such that for any hypersphere Σ of radius r with center in a point of T the inequality $J_{m,p}[u,T\cdot\Sigma]\leqq Hr^\nu$ is satisfied for all u, then, provided $\nu>(n-mp)/m$, the family U is compact with respect to uniform convergence and satisfies a uniform Hölder condition with exponent $\lambda=\nu+(mp-n)/m$ if $\lambda<1$ and with any exponent <1 if $\lambda\geqq 1$. He states that this result is easily extended to the case where the functions u, instead of being p-regular, have derivatives of the first $(p-1)$ orders which are absolutely continuous in the sense of Tonelli. Then evidently they have almost everywhere partial derivatives

of order p; these must be taken as before to be summable with exponent $m \geqq 1$. This result, for $p=1$ and $m=1$, yields a theorem of Morrey [Univ. California Publ. Math. (N.S.) 1, 1–130 (1943), see p. 15, Theorem 22; these Rev. 6, 180] and, for $n=2$, $p=1$, yields a result of Tonelli [Ann. Scuola Norm. Super. Pisa (2) 2, 89–130 (1933), p. 97].

J. M. Danskin (Santa Monica, Calif.).

Stampacchia, Guido. Sopra una classe di funzioni in n variabili. Ricerche Mat. 1, 27–54 (1952).

The author proves theorems in n dimensions analogous to those proved in an earlier paper [Giorn. Mat. Battaglini (4) 3(79), 169–208 (1950); these Rev. 13, 955] for two dimensions. He considers a class of functions which coincides with a class considered by Calkin [Duke Math. J. 6, 170–186 (1940); these Rev. 1, 208] and Morrey [ibid. 6, 187–215 (1940); these Rev. 1, 209] and proves theorems of the following type: "If the functions $f(P)$ are restricted in a certain sense on their domains, then from the equiboundedness of the integrals $\int \sum_i |\partial f/\partial x_i|^a dx_1 \cdots dx_m$ one may infer that a certain subsequence converges quasi-uniformly in the sense that the convergence is uniform except on a set of points whose projections on the $(m-r)$-dimensional coordinate spaces are sufficiently small in measure, where $r<\alpha$." The author points out that he has found the proofs of two theorems on compactness in the earlier paper "unsatisfactory" and corrects them here. *J. M. Danskin.*

Kober, H. Note on the extension of rectangle functions. Duke Math. J. 19, 409–416 (1952).

Given a basic closed rectangle $R_0=(0 \leqq x \leqq a, 0 \leqq y \leqq b)$, let $[R]$ be the collection of all rectangles $R \subset R_0$ with sides parallel to the sides of R_0 and $f(R)$, $R\varepsilon[R]$, a rectangle function. Reichelderfer, Rechard, and Ringenberg [same J. 8, 231–242 (1942); 15, 151–158 (1948); Trans. Amer. Math. Soc. 61, 134–146 (1947); these Rev. 3, 74; 9, 418; 8, 320] have given various conditions in order that $f(R)$ can be extended to an additive function $\phi(B)$ defined for all Borel sets $B \subset R_0$. In the cited papers a condition of continuity was used, namely, $f(R_\epsilon) \to f(R)$ as $\epsilon \to 0$ where R_ϵ is the rectangle containing R with sides parallel to and at a distance ϵ from those of R. In the present paper also, a condition of weak continuity on a figure P is used, namely, that given $\epsilon>0$, there is a $\delta>0$ such that $|f(R)|<\epsilon$ whenever R has diameter $<\delta$ and $R \subset P$. By figure P is meant a finite sum of closed rectangles. Finally, the following condition C is introduced: (i) $f(R)$ is continuous on R_0^0, i.e., on any $R \subset R_0^0$ where R_0^0 is the set of the interior points of R_0; (ii) $f(R)$ is weakly continuous on any $P \subset R_0$ which does not contain the vertices of R. The following statements among others proved by the author add further information on the question whether $f(R)$ can be extended. I) If $f(R)$ admits an extension and satisfies C, then $f(R)$ is additive and of bounded variation; II) If $f(R)$ is additive, then $f(R)$ can be extended if and only if f satisfies condition C and is of bounded variation; III) If $f(R)$ is additive, then $f(R)$ can be extended if and only if its total variation can be extended.

L. Cesari (Lafayette, Ind.).

Radó, T., and Reichelderfer, P. V. On generalized Lipschitzian transformations. Rivista Mat. Univ. Parma 2, 289–301 (1951).

Let T: $Q \to M$ be a continuous mapping from the unit square in E_2 to a metric space M. The mapping T is said to be a generalized Lipschitzian transformation if there exists a real-valued, non-negative, summable function Φ on Q such that if S is any oriented square in Q (a square with sides parallel to the axes), $\omega(S, T) \leqq \{\int\int_S \Phi(u, v)dudv\}^{1/2}$ where $\omega(S, T)$ is the variation of T over S. The authors discuss the relationships of this with various other similar generalizations of the Lipschitz condition and with various notions of absolute continuity. In particular, it is shown that if T is generalized Lipschitzian from Q to E_2, then T is absolutely continuous in the sense of Tonelli (ACT) and in the sense of Banach (ACB). The latter result gives a generalization of a theorem of Cesari [Ann. Mat. Pura Appl. (4) 21, 157–188 (1942); these Rev. 6, 43]. *R. E. Fullerton.*

Kudryavcev, L. D., and Kaščenko, Yu. D. On change of variable in an integral. Doklady Akad. Nauk SSSR (N.S.) 84, 869–871 (1952). (Russian)

In the case of a one-to-one differentiable change of variables of Jacobian J, the authors establish the transformation formula for the multiple Lebesgue integral of $f \cdot J$ over an n-dimensional Euclidean domain, subject to the assumption that f and J belong to L^2. Very much more general results have been announced by Radó and Reichelderfer [Proc. Nat. Acad. Sci. U. S. A. 35, 678–681 (1949); these Rev. 11, 588]. *L. C. Young* (Madison, Wis.).

Glivenko, E. V. On planar variation. Mat. Sbornik N.S. 30(72), 581–600 (1952). (Russian)

Let $f(x, y)$ be a real-valued function of two real variables x, y that is defined and continuous on the unit square I $[0, 1; 0, 1]$, and let M be a Borel set contained in I. A. S. Kronrod [Uspehi Matem. Nauk 5, no. 1(35), 24–134 (1950), especially pp. 87–90; these Rev. 11, 648] defined the planar variation $W(f, M)$ of f on M to be the Lebesgue integral $\int_{-\infty}^{\infty} \nu(f, M, E_t)dt$, where $\nu(f, M, E_t)$ is the Hausdorff linear measure of the subset E_t of M on which $f(x, y)=t$, and he established the following properties of $W(f, M)$: (1) $W(f, M) \geqq 0$; (2) $W(cf, M)=|c|W(f, M)$ for any constant c; (3) $W(f, M)$ is a completely additive function of Borel sets M; (4) $W(f+c, M)=W(f, M)$ for any constant c; (5) $W(f, M)$ is a lower semi-continuous function of f; (6) if M' is the image of M, and (x', y') that of (x, y), under a congruence operation, then $W(f(x', y'), M')=W(f(x, y), M)$; (7) if $f(x, y)=y$, then $W(f, I)=1$. The author shows that properties (1)–(7) suffice to determine $W(f, M)$ uniquely, and could thus be the basis of an alternative (axiomatic) definition. He proves, incidentally, that there is always a sequence of polyhedra $z=f_n(x, y)$ such that $f_n(x, y) \to f(x, y)$ uniformly on I and $W(f_n, I) \to W(f, I)$ as $n \to \infty$.

H. P. Mulholland (Birmingham).

Theory of Functions of Complex Variables

Hayes, N. D. The roots of the equation $x=(c \exp)^n x$ and the cycles of the substitution $(x|ce^x)$. Quart. J. Math., Oxford Ser. (2) 3, 81–90 (1952).

A cycle of the substitution $(x|ce^x)$ is a sequence $\{x_1, x_2, \cdots, x_n\}$ of complex numbers such that $x_2=ce^{x_1}$, $x_3=ce^{x_2}$, $\cdots$, $x_n=ce^{x_{n-1}}$, and $x_1=ce^{x_n}$. Evidently any of these numbers is a root of the equation $x=(c \exp)^n x$. The author's principal result is as follows: Except in the cases mentioned below, there exists for each set of integers $(\geqq 0)$ $(h_1, \cdots, h_n)$ a cycle $\{x_1, \cdots, x_n\}$ of the substitution $(x|ce^x)$, and the x_i satisfy $2h_i\pi < \operatorname{Im} x_i < 2(h_i+1)\pi$. The only exceptions and additional roots are (i) where $c<0$, the equation has a

negative real root; (ii) when $c \leqq -e$ and n is even, there are two additional negative roots and there is no complex root corresponding to the set $(0, -1, 0, \cdots, -1)$ or to $(-1, 0, -1, \cdots, 0)$; (iii) when $0 < c < e^{-1}$ there are two real roots and there is no complex root for the set $(0, 0, \cdots, 0)$ or the set $(-1, -1, -1, \cdots, -1)$. Here c is assumed real. If c is unreal the statement may be simplified. This work is of interest in the study of the asymptotic behavior of

$$a^{a^{a^{\cdots}}}$$

as the number of a's is increased. *J. M. Danskin.*

Sinclair, Annette. The zeros of an analytic function of arbitrarily rapid growth. Proc. Amer. Math. Soc. **3**, 416–424 (1952).

Generalising results due to Poincaré and Borel, P. W. Ketchum [Trans. Amer. Math. Soc. **49**, 211–228 (1941); these Rev. **2**, 273] has investigated point sets F such that, for any given $G(z) \geqq 0$, there exists an $f(z)$, analytic except where $G(z)$ is unbounded, with $|f(z)| \geqq G(z)$ on F (Theorem IV). He proposed the additional restriction: $f(z) \neq 0$ except at certain points of the complementary set $C(F)$. This problem is treated by the author. The definitions and notations are the same as in a previous paper [ibid. **72**, 148–164 (1952); these Rev. **13**, 832]; again, given a Q-set S, $E \subset C(S)$ is said to be an E_S-set when it is closed and contains all the sequential limit (S.-s.l.) points of S, i.e., of the components S_j $(j = 1, 2, \cdots)$ of S; and $M(z)$ is $A^*(E)$ bounded on S if there is an $f(z)$, $\neq 0$ and analytic on $C(E)$, such that $|f(z)| > |M(z)|$ on S. For the definitions of a Q_R-set and E-free nested components the reader must be referred to the paper. If S is a Q-set, and if any $B^*(S)$ and numbers M_j, N_j are fixed $(0 \leqq N_j < M_j)$, then there is an $f(z)$, analytic and $\neq 0$ on $C(B^*)$, and such that $N_j < |f(z)| < M_j$ on S_j $(j = 1, 2, \cdots)$. This is deduced as a corollary of Theorem 5 of the previous paper. It follows that when S does not separate the plane and has $z = \infty$ as only S.-s.l. point, then there is an $f(z)$, $|f(z)| > N_j$ on S_j, which is a non-vanishing entire function. The main result is Theorem 2: Let S be a Q_R-set, E an E_S-set; if, and only if, S has no E-free nested sequence of components, then every $M(z)$ that is bounded on each S_j is $A^*(E)$ bounded on S. The condition is satisfied, for instance, if S consists of a sequence of nested rings and E is a $B^*(S)$-set, or contains some $B^*(S)$-set. *H. Kober.*

Wolibner, W. Sur une relation entre les singularités des fonctions analytiques. Colloquium Math. **2** (1951), 182–185 (1952).

Let $f(z)$ be a uniform analytic function or a branch of a multiform function, A its region of existence and S the boundary of A. Suppose that $f(z)$ is regular at $z = 0$ and $z = \infty$, let $f(\infty) = 0$ and $f(z) = \sum_0^\infty a_n z^n$ near $z = 0$. Then there exists a polygonal line L, contained in A, joining 0 to infinity. Call G_L the z-plane cut by L and put $H_L = \alpha(G_L)$, where $x = \alpha(z) = -1/\lg z$, where $\lg z$ denotes any branch of $\log z$ given in G_L. Putting $K = \alpha(S)$, denote by B that component of the complement of K in the x-plane which contains $x = 0$. The author proves that there exists a sequence (b_k) such that $a_n = \sum_0^\infty b_k n^k/k!$ $(n = 0, 1, 2, \cdots)$ and that the analytic function defined by the element $\varphi(x) = \sum_0^\infty b_k x^k$ is regular inside B. Now let $\varphi(x)$ be a uniform analytic function or a branch of a multiform function, D its region of existence and F the boundary of D. Suppose that $\varphi(x)$ is regular at $x = 0$ and that $\varphi(x) = \sum_0^\infty b_k x^k$. Put $E = \beta(F)$, where $z = \beta(x) = e^{-1/x}$, and denote by C that component of the complement of E in the z-plane which includes the point 0, this point not belonging to E. The author proves that the analytic function defined near 0 by the element $f(z) = \sum_0^\infty a_n z^n$, where $a_n = \sum_0^\infty b_k n^k/k!$, is regular inside C save at $z = 1$ where it has a simple pole. *R. Wilson* (Swansea).

Sunyer i Balaguer, F. Sur la substitution d'une valeur exceptionnelle par une propriété lacunaire. Acta Math. **87**, 17–31 (1952).

Continuing the discussion and extension [Mem. Real Acad. Ci. Art. Barcelona (3) **29**, 475–516 (1948); Collectanea Math. **2**, 129–174 (1949); these Rev. **12**, 88, 489] of results previously announced [C. R. Acad. Sci. Paris **224**, 1609–1610; **225**, 21–23 (1947); these Rev. **9**, 22] the present paper deals in particular (chapter 3) with power series of the form $F(z) = \sum a_n z^{\lambda_n}$ of radius of convergence unity giving analogues of the theorems of Landau and Schottky in which gap conditions imposed on the λ_n form part of the hypothesis. For instance (the sixth of nine theorems) if the number of zeros $n(r)$ of $F(r)$ in $|z| < r < 1$ satisfies $n(r) < q r (1-r)^{-\lambda}$, then the maximum modulus $M(r)$ of $F(z)$ on $|z| = r$ satisfies $\log M(r) < A(1-r)^{-\rho} + B$ for $r > r_0 < 1$, where A, B, r_0 depend in a manner partly determined on q, λ, and the gap hypotheses. Chapters 1 (finite order) and 2 (infinite order) give similar results for integral functions.

A. J. Macintyre (Aberdeen).

Mandelbrojt, S. Quelques théorèmes de composition. J. Math. Pures Appl. (9) **31**, 79–90 (1952).

L'auteur étend un résultat obtenu par lui en collaboration avec H. Brunk [Duke Math. J. **18**, 297–306 (1951); ces Rev. **12**, 808] par l'étude du produit de composition $\int_c f(s)\varphi(w-s)ds$ et obtient une proposition notablement plus générale. (I) Soient $F(z)$, $\phi(z)$, $z = x+iy$, deux fonctions holomorphes pour $x > 0$, continues pour $x \geqq 0$ avec $|F(z)| = O(|z|^\delta)$, $|\phi(z)| = O(|z|^\delta)$, $\delta > 0$, quand $z \to 0$. (II) Soient deux suites $\{M_n\}$, $\{M'_n\}$, de nombres positifs, $\log M_n$, $\log M'_n$ étant fonctions convexes de n; si l'on a les approximations pour $x > 0$, et $n \geqq 1$:

$$(1) \qquad \left|F(z) - \sum_1^n a_\nu z^{-\nu}\right| < M_n z^{-n}, \quad \left|\phi(z) - \sum_1^n b_\nu z^{-\nu}\right| < M'_n z^{-n},$$

avec $a_{2\nu-1} = b_{2\nu-1} = 0$ pour $\nu \geqq 1$, $a_{\nu_n} b_{\nu_n} = 0$, pour $n \geqq 1$, $\{\nu_n\}$ étant une suite d'entiers contenant tous les impairs pour laquelle on définit $(t) D = N(t)/t$, $N(t)$ étant le nombre des ν_n inférieurs à t, $D.(t) = \inf_{x \geq t} D(x)$, puis à partir de la fonction $C(\sigma) = \sup_{n \geq 1} [n\sigma - \log (M_n M'_n)]$ la quantité $D.[C(\tau)]$ en prenant $t = C(\tau)$ et l'intégrale

$$I = \int_0^\infty C(\sigma) \exp\left[-\int^\tau \frac{d\tau}{2D.[C(\tau)]-1}\right] d\sigma.$$

(III) Alors si la densité inférieure $D. = \lim_{t \to \infty} D.(t)$ surpasse $\frac{1}{2}$ et si l'on a $I = \infty$, l'une des deux fonctions F ou ϕ est identiquement nulle. L'intégrale I est introduite par un résultat précédent de l'auteur [Ann. Sci. École Norm. Sup. (3) **63**, 351–378 (1947); ces Rev. **9**, 229]. Le lemme suivant: si q_n est une suite d'entiers avec $\sum q_n^{-1} < \infty$, il existe $f(x)$, $f \not\equiv 0$, $-\infty < x < \infty$, avec $f(0) = f^{(q_n)}(0) = 0$, $n \geqq 1$, $f^{(m)}(0) \neq 0$ pour $m \neq q_n$, et avec $|f^{(n)}(x)| \leqq 1$, $n \geqq 0$, pour $-\infty < x < \infty$, permet ensuite d'établir un théorème d'unicité [$F \equiv 0$ sous des conditions (1)] et un résultat de sens inverse.

P. Lelong (Lille).

Mitrović, Dragiša. Une généralisation du théorème de Rouché. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **7**, 19–22 (1952). (Serbo-Croatian. French summary)

Let C be a simple closed curve, and D the domain bounded by C together with its boundary. Let $h(z)$ be holomorphic in D, $\varphi(z)$ and $f(z)$ meromorphic in D, and suppose $|\varphi(z)|<|f(z)|$ on C. Then

$$\frac{1}{2\pi i}\int_C h'(z)\log\left[1+\frac{\varphi(z)}{f(z)}\right]dz = \sum_1^n \alpha_\nu h(a_\nu)-\sum_1^k \alpha_j' h(a_j')+\sum_1^m \beta_\tau' h(b_\tau')-\sum_1^p \beta_\mu h(b_\mu),$$

where a_ν, b_μ, are the zeros and poles of $f(z)$ of order α_ν, β_μ, and a_j', b_τ' are the zeros and poles of $f(z)+\varphi(z)$ of order α_j', β_τ'. The special case $h(z)\equiv 1$, $f(z)$ and $\varphi(z)$ holomorphic, gives Rouché's theorem. *A. W. Goodman.*

Nassif, M. On the behaviour of the function

$$f(z)=\sum_{n=0}^{\infty} e^{\sqrt{2}\pi i n^2}\frac{z^{2n}}{n!}.$$

Proc. London Math. Soc. (2) **54**, 201–214 (1952).

Es wird die ganze Funktion $f(z)=\sum_{n=0}^{\infty}\exp(i\alpha\pi n^2)z^{2n}/n!$ $(\alpha=\sqrt{2})$ untersucht. Das Interesse richtet sich darauf, zu sehen, was durch Beifügen der Faktoren $\exp i\alpha\pi n^2$ in der Potenzreihe von $\exp z^2$ bewirkt wird. Mit

$$z_N=iN^{1/2}\exp\tfrac{1}{2}\pi i\alpha(1-2N)$$

hat $f(z)$ in jedem Kreis $C_N(|z\pm z_N|<N^{-1/2}/4)$ genau eine Nullstelle und ausserhalb kleiner Kreislein um diese Nullstellen ist $\log|f(z)|>(\frac{1}{2}+o(1))|z|$, $z\to\infty$. Zum Beweis verwendet Verf. einige für seine Zwecke hergeleitete Resultate über die Thetafunktion $\vartheta_1(v,\tau)$ und ihre Ableitungen [für $v=0$ vgl. Hardy und Littlewood, Acta Math. **37**, 193–239 (1914)]. *A. Pfluger* (Zürich).

Tims, S. R. Note on a paper by M. Nassif. Proc. London Math. Soc. (2) **54**, 215–218 (1952).

Im Anschluss an die vorangehende Arbeit [vgl. voranstehendes Referat] wird gezeigt, dass die z_N asymptotisch lokal bis auf einen Fehler $o(1)$ durch ein Gitter approximiert werden können. *A. Pfluger* (Zürich).

Delange, Hubert. Un théorème sur les fonctions entières à zéros réels et négatifs. J. Math. Pures Appl. (9) **31**, 55–78 (1952).

Let $f(z)$ be an integral function with real negative zeros and genus p. Let $n(t)$ be the number of these zeros in $(-t,0)$ and suppose $|\theta|<\pi$, $p<\rho<p+1$, $\cos\rho\theta\neq 0$. The author establishes the following results. If for an odd integer m we have (1) $m\pi/2(\rho+1)<|\theta|<m\pi/2\rho$, we may have (2) $\log|f(re^{i\theta})|\sim(-1)^p A r^\rho\cos\rho\theta$ without having (3) $n(t)=O(t^\rho)$. However (2) and (3) together imply (4) $n(t)\sim A|\sin\pi\rho/\pi|t^\rho$. On the other hand if θ does not lie in the ranges (1), then (2) alone is sufficient to imply (4). A more general result announced previously by the author [C. R. Acad. Sci. Paris **225**, 483–485 (1947); these Rev. **9**, 140] with ρ replaced by approximate order $\rho(r)$ could be proved by the same method. The case $\theta=0$ of this latter result is due to Valiron [Ann. Fac. Sci. Univ. Toulouse (3) **5**, 117–257 (1913)]. *W. K. Hayman* (Exeter).

***Nikol'skiĭ, S. M. Inequalities for entire functions of finite degree and their application in the theory of differentiable functions of several variables.** Trudy Mat. Inst. Steklov., v. 38, pp. 244–278. Izdat. Akad. Nauk SSSR, Moscow, 1951. (Russian) 20 rubles.

This paper contains L^p and l^p inequalities for entire functions of exponential type in several variables, theorems on the approximation of functions by entire functions of exponential type, and applications to inclusion relations between symmetric Lipschitz classes ("smooth" functions), extending theorems of Kondrašov [C. R. (Doklady) Acad. Sci. URSS **48**, 535–538 (1945); these Rev. **8**, 32] and Sobolev [Mat. Sbornik N.S. **4(46)**, 471–497 (1938)]. The first part overlaps work of Plancherel and Pólya [Comment. Math. Helv. **9**, 224–248; **10**, 110–163 (1937)], but the methods are different and the constants in the inequalities are better. For the sake of typographical simplicity the results quoted in this review will be stated whenever possible in terms of one or two complex variables $(z=x+iy, w=u+iv)$, although the author proves them all for functions of n variables.

Let $g(z,w)$ be an entire function of exponential type (finite degree) μ in z, ν in w, with $\{\|g\|_p\}^p=\int\int|g(x,u)|^p dxdu$, integrals over $(-\infty,\infty)$; correspondingly for $g(z)$. The first problem is to estimate a sum $\sum|g(x_k)|^p$ in terms of $\|g\|_p$, the result being that if $k\alpha/\mu\leqq x_k\leqq(k+r)\alpha/\mu$, then $\alpha\sum|g(x_k)|^p\leqq\mu(1+r\alpha)^p\|g\|_p^p$. The proof depends on the inequality $\|g'\|_p\leqq\mu\|g\|_p$, which the author deduces from an interpolation formula. The next main result is that if $p'>p$, then $\|g\|_{p'}\leqq 2\mu^{1/p-1/p'}\|g\|_p$ (for n variables, 2 becomes 2^n; μ becomes $\mu\nu$ for $n=2$, and so on); the proof depends on inequalities which connect $\|g\|_p$ with

$$\sup_{t_l}\{\sum\sum g(x_k-t_1,u_l-t_2)\},$$

where (k,l) ranges over all pairs of integers and u_l is defined analogously to x_k. The second section gives analogous results for trigonometric polynomials with integrals now extended over a period [analogues of results of Marcinkiewicz and Zygmund, Fund. Math. **28**, 131–166 (1937)].

In the third section the author generalizes Zygmund's smooth functions [Duke Math. J. **12**, 47–76 (1945); these Rev. **7**, 60] to n variables and nonperiodic functions. His functions of class $H_p^{(r,s)}$ satisfy

$$\left\|\frac{\partial^\rho f(x+h,u)}{\partial x^\rho}-2\frac{\partial^\rho f(x,u)}{\partial x^\rho}+\frac{\partial^\rho f(x-h,u)}{\partial x^\rho}\right\|_p\leqq M|h|^\alpha,$$

where $r=\rho+\alpha$, ρ is an integer, and $0<\alpha\leqq 1$, and the corresponding inequality with derivatives with respect to u and index $\sigma=s-\beta$. Results of S. Bernstein [Doklady Akad. Nauk SSSR **57**, 111–114 (1947); these Rev. **9**, 235] are generalized to several variables. The paper contains more detailed and precise results than can be reproduced here, but the general situation is that the best pth power approximation $A_{\mu,\nu}(f)_p$ by entire functions of exponential types μ, ν is, if f belongs to $H_p^{(r,s)}$, at least of order $\mu^{-r}+\nu^{-s}$; while if f admits such an approximation for a suitable sequence of large μ and ν, then f can be approximated by an entire function g of exponential types 1, 1, in such a way that $f-g$ belongs to $H_p^{(r,s)}$. Furthermore, the set of functions f of L^p of uniformly bounded norm, with $\sup_f A_{\mu,\nu}(f)_p$ approaching zero as μ and ν become infinite, is (sequentially) compact in the L^p-norm on bounded sets.

In section 4 the results on approximation are applied to show that a function belonging to $H_p^{(r_1,\cdots,r_n)}$ also belongs with respect to its first m variables to $H_{p'}^{(s_1,\cdots,s_m)}$, where

$p'>p$ and $\rho_j=r_j\kappa$, provided

$$\kappa=1-(p^{-1}-p'^{-1})\sum_{j=1}^{m}r_j^{-1}-p'^{-1}\sum_{j=m+1}^{n}r_j^{-1}>0.$$

The theory is illustrated by examples. A theorem of the author [Doklady Akad. Nauk SSSR 59, 1533–1536 (1948); these Rev. 9, 427] is generalized from $p=\infty$ to $p\geqq 1$, and a theorem is given on the sequential compactness of a set of functions belonging to $H_p^{(r_1,\cdots,r_n)}$. *R. P. Boas, Jr.*

Singh, S. K. A note on entire functions. J. Univ. Bombay (N.S.) **20**, part 5, sect. A, 1–7 (1952).

The author proves several theorems on entire functions; the following are representative. (i) Let $N^{(n)}(r)$ be the maximum modulus of $f^{(n)}(z)$; if $f^{(n)}(0)\geqq 0$ for all n, and $R>r$, then $(r/R)^n M^{(n)}(r)/M^{(n)}(R)$ decreases as n increases. (ii) If $\mu(r)$ is the maximum term in the power series for $f(z)$, then

$$\{\mu(r_1)^{m_1}\mu(r_2)^{m_2}\}^{1/(m_1+m_2)}\geqq\mu\{[r_1^{m_1}r_2^{m_2}]^{1/(m_1+m_2)}\}.$$

(iii) If $f(z)$ is of order ρ and lower order λ, then the lim sup and lim inf of $\log\{r\mu'(r)/\mu(r)\}/\log r$ are respectively ρ and λ. (iv) If $M(r)$ and $m(r)$ are the maximum and minimum moduli, $n(r)$ is the number of zeros in $|z|\leqq r$, $R\geqq r>0$, and $|f(0)|=1$, then

$$M(R)/m(r)\geqq(R/r)^{n(r)},\quad m(R)/M(r)\leqq(R/r)^{n(R)}.$$

(v) If the exponents of convergence for both the simple a-points and simple b-points are less than the order, there must exist both multiple a- and b-points.

R. P. Boas, Jr. (Evanston, Ill.).

Šteĭnberg, N. S. On the interpolation of entire functions. Mat. Sbornik N.S. **30(72)**, 559–574 (1952). (Russian)

The first four theorems of this paper deal with the Newton interpolation series and were announced in Doklady Akad. Nauk SSSR **71**, 21–22 (1950); these Rev. 11, 510. The fifth sharpens a result of Ibragimov [Mat. Sbornik **21(63)**, 49–62 (1947); these Rev. 9, 84] on the Abel-Gončarov interpolation series [defined in the cited review] as follows: if there is a positive function $\lambda(r)$ such that $r\lambda(r)\to\infty$, $\frac{1}{2}r-r\lambda(r)$ increases and $\limsup\{r\log M(r)\}/n\{r(\frac{1}{2}-\lambda(r))\}<\infty$, then the series converges to the function generating it; here $n(r)$ is the number of interpolation points in $|z|\leqq r$.

R. P. Boas, Jr. (Evanston, Ill.).

Džrbašyan, M. M. Uniqueness and representation theorems for analytic functions. Izvestiya Akad. Nauk SSSR. Ser. Mat. **16**, 225–252 (1952). (Russian)

The author deals with the uniqueness and representation problems associated with the data $f^{(n)}(a_n)$ and the Abel-Gončarov interpolation series [cf. the preceding review]. Let $A_\lambda(\rho,\sigma)$ denote the class of entire functions of growth less than order ρ, type σ, with $f^{(k)}(0)=0$ except when k belongs to a sequence $\{\lambda_n\}$ of integers; let $\mu_n\leqq\lambda_n<\mu_{n+1}$. By assuming to begin with that some derivatives vanish at the origin, the author can improve known results on the uniqueness problem. His first theorem gives a sufficient condition for the conditions $f^{(\mu_n)}(a_n)=0$ and $f(z)\in A_\lambda(\rho,\sigma)$ to imply $f(z)\equiv 0$; the next two theorems are devoted to examples showing that the sufficient condition cannot "in general" be improved. Then the author shows that the sufficient condition also ensures the representation of $f(z)$ in the form $\sum f^{(\mu_n)}(a_n)Q_n(z)$, where Q_n are the Gončarov polynomials based on the points a_n. The theory is illustrated by some special cases. Thus if $\lambda_n=n$, the condition

$$\limsup k^{1-1/\rho}|a_k|<(\sigma\rho)^{-1/\rho}\log 2$$

is sufficient for $f(z)\equiv 0$ if $f^{(n)}(a_n)=0$ and $f(z)\in A_\lambda(\rho,\sigma)$. If $\lambda_{k+1}-\lambda_k\to\infty$ and $\rho\leqq 1$, then $f(z)\equiv 0$ if $f^{(\mu_n)}(a_n)=0$ and $\limsup|a_k|\lambda_{k+1}^{1-1/\rho}(\lambda_{k+1}-\mu_k)^{-1}<(\sigma\rho e)^{-1/\rho}$. Further results of this character are given. Finally the author gives similar results for functions analytic in a finite circle.

R. P. Boas, Jr. (Evanston, Ill.).

Džrbašyan, M. M. On the integral representation and uniqueness of some classes of entire functions. Doklady Akad. Nauk SSSR (N.S.) **85**, 29–32 (1952). (Russian)

A famous theorem of Paley and Wiener [Fourier transforms in the complex domain, Amer. Math. Soc. Colloq. Publ., v. 19, New York, 1934, p. 13] states that an entire function of order $\rho=1$ and type σ, belonging to L^2 on the real axis, admits the representation $f(z)=\int_{-\sigma}^{\sigma}e^{izt}\phi(t)dt$, with $\phi(t)$ in L^2. The author states without proof several theorems generalizing this to other orders ρ and gives some applications. The representations involve the Mittag-Leffler functions $E_\rho(z)=\sum z^n/\Gamma(1+n/\rho)$. In the first the hypothesis is $\rho\geqq\frac{1}{2}$ and $\int_0^\infty|f(te^{-i\theta})|^2t^{\rho-1}dt<\infty$, and the representation is too complicated to quote here. In the second, $\frac{1}{2}\leqq\rho<1$, $\int_0^\infty|f(t\exp\{\pm\pi(1-1/2\rho)i\})|^2t^{\rho-1}dt<\infty$, and the conclusion is

$$f(z)=\int_0^{\sigma^{1/\rho}}E_\rho(-zv)\psi(v)v^{\rho-1}dv,\quad\int_0^{\sigma^{1/\rho}}|\psi(v)|^2v^{\rho-1}dv<\infty.$$

In the third, $1\leqq\rho<2$, the hypothesis is the convergence of

$$\int_0^\infty|f(t\exp\{\pm\tfrac{1}{2}\pi(1\pm 1/\rho)i\})|^2t^{\rho-1}dt,$$

and the conclusion is

$$f(z)=\int_{-\sigma^{1/\rho}}^{\sigma^{1/\rho}}E_\rho(ivz)\psi(v)v^{\rho-1}dv,$$

with the corresponding integrability condition on $\psi(v)$. In the fourth, $f(z)$ is of integral order $p\geqq 1$, $\int_0^\infty|f(z)|^2|z|^{p-1}|dz|$ converges on $2p$ equally spaced rays, and $f(z)$ is represented as a sum of p integrals. The applications are uniqueness theorems inferring $f(z)\equiv 0$ from $f^{(\lambda_n)}(0)=0$ for suitable sequences $\{\lambda_n\}$ and growth conditions on $f(z)$; they presumably result from transforming the hypothesis, via an integral representation, into one involving an analytic function with zeros λ_n. *R. P. Boas, Jr.* (Evanston, Ill.).

Dugué, Daniel. Théorème d'impossibilité relatif aux fonctions elliptiques analogue à un théorème de M. Borel sur les exponentielles. C. R. Acad. Sci. Paris **233**, 1566–1567 (1951).

Auf Grund des Satzes von R. Nevanlinna, dass eine meromorphe Funktion höchstens vier vollständig verzweigte Werte besitzen kann, gelangt Verf. zu der Bemerkung, dass zwischen ganzen Funktionen H_i und der Weierstrassschen elliptischen Funktion $\wp$ keine Relation von der Form $A\wp(H_1)+B\wp(H_2)=1$ (A und B = konst.) und dass die Relation $A\wp(H_1)+B\wp(H_2)=0$ nur für spezielle A und B bzw. $\wp$ bestehen kann. *A. Pfluger* (Zürich).

Dugué, D. Fonctions fuchsiennes et familles normales. J. Math. Pures Appl. (9) **31**, 19–35 (1952).

The classical proof of Picard's theorem employing the modular function is modified by using instead several other automorphic functions whose properties are discussed from first principles. A typical result is Theorem 4. If $f(z)$ is analytic and uniform in a part of the complex plane (or the whole plane) in which it takes all values infinitely often, is

without asymptotic values, and has no values of multiplicity greater than two, then there are at least four values of a such that $f(z)=a$ has some double roots. Similar generalisations of Picard's theorem on uniformisation of algebraic functions are given. *A. J. Macintyre* (Aberdeen).

Itô, Jun-iti. On the function whose imaginary part on the unit circle changes its sign $2p$ times. Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. **4**, 107–114 (1944).

The author studies the class of functions

$$f(z)=z^p+c_{p+1}z^{p+1}+\cdots$$

which are continuous in the closed unit circle and whose imaginary parts change sign at the $2p$ points $e^{i\sigma_s}$, $s=1, \cdots, 2p$, $0\leqq\sigma_1<\sigma_2<\cdots<\sigma_{2p}\leqq 2\pi$. As did the reviewer [Ann. of Math. **38**, 770–783 (1937)], he characterizes $f(z)$ in terms of a function $F(z)$ regular for $|z|<1$ and continuous with positive real part in $|z|\leqq 1$, $F(0)=1$, from which bounds for the modulus of $f(z)$ and the coefficients c_n, $n\geqq p+1$, are obtained. By using the Poisson integral representation for $F(z)$, the author obtains the following new and more explicit result that $f(z)$, $|z|<1$, lies within the smallest open convex domain containing the curve

$$W=W(\theta)=\frac{(e^{i(\mu+\theta)}-z)z^p}{(e^{i\theta}-z)\prod_{s=1}^{2p}(e^{i\sigma_s}-z)},\quad 0\leqq\theta\leqq 2\pi,\quad \mu=\sum_1^{2p}\sigma_s,$$

while the coefficient c_{p+n} lies in an analogous smallest open convex domain. *M. S. Robertson.*

Valiron, Georges. Sur une classe de fonctions algebroïdes d'ordre nul. Rend. Circ. Mat. Palermo (2) **1**, 63–70 (1952).

The functions considered are solutions of the equation $A_0(z)+A_1(z)u+\cdots+A_\nu(z)u^\nu$ where the $A_k(z)=\sum c_{kn}z^n$ are integral functions without common zeros satisfying

$$\log|A_k(z)|=O\{(\log|z|)^2\}.$$

It is shown that the Nevanlinna characteristic $T(r, u)$ is determined asymptotically by the moduli of the c_{kn}. If $A_\nu(z)=1$, $u(z)$ may have asymptotic paths even when the $A_k(z)$ are polynomials but the greatest determination $U(z)$ of the ν values of $|u|$ satisfies $\log U(z)>(1-\epsilon)T(|z|, u)$ for sufficiently large r and all z excepting a set of arbitrarily small measure values of $\arg z$. The second result is extended to wider classes of functions. *A. J. Macintyre.*

Lohwater, A. J. The boundary values of a class of meromorphic functions. Duke Math. J. **19**, 243–252 (1952).

Zunächst beweist Verf. diesen auch an sich interessanten Satz: Genügt die in $|z|<1$ harmonische Funktion $u(r, \varphi)$ der Bedingung $\int_0^{2\pi}|u(r, \varphi)|\,d\varphi<M$, $r<1$, und sind ihre Randwerte $\lim_{r\to 1}u(r, \varphi)$ überall, wo sie existieren, endlich und fast überall null, so ist u die Konstante 0. Mit Hilfe der Nevanlinnaschen Integraldarstellung beschränktartiger Funktionen folgt daraus: Eine in $|z|<1$ beschränktartige meromorphe Funktion mit nur endlich vielen Nullstellen und Polen, die keine rationale Funktion ist, wo die radialen Grenzwerte $\lim_{r\to 1}|f(re^{i\varphi})|$ fast überall den Wert 1 haben, hat sicher auch einen radialen Grenzwert 0 oder ∞. Dies verallgemeinert ein Ergebnis von W. Seidel [Trans. Amer. Math. Soc. **36**, 201–226 (1934)] und Hössjer und Frostman [Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **3**, no. 16, 1–8 (1933)]. Der genannte Satz bleibt auch noch für nichtbeschränktartige Funktionen richtig, wenn unter sonst gleichen Voraussetzungen in der Behauptung radiale Annäherung durch die Annäherung an einen Randpunkt längs irgend eines Jordanbogens ersetzt wird. Zum Beweis genügt es Funktionen ohne Nullstellen und Pole zu betrachten. Nun ist $\inf_{|z|<1}|f(z)|=0$, da sonst $1/f(z)$ beschränkt wäre. Daraus folgt die Existenz eines in einem Randpunkt endenden Jordanbogens, längs welchem f gegen null strebt, nach Ansicht des Ref. aber auf wesentlich einfachere Art als durch den Verf. ausgeführt wird. Weitere Ergebnisse stehen im Zusammenhang mit Untersuchungen von C. Carathéodory [Comment. Math. Helv. **19**, 263–278 (1946); diese Rev. **8**, 508] und Frostman [Dissertation, Lund, 1935]. *A. Pfluger* (Zürich).

Tsuji, Masatsugu. On the order of the derivative of a meromorphic function. Tôhoku Math. J. (2) **3**, 282–284 (1951).

The author gives a simple proof of a theorem of G. Valiron [see Acta Math. **47**, 117–142 (1926); also J. M. Whittaker, J. London Math. Soc. **11**, 82–87 (1936)] that, if $f(z)$ is meromorphic in the plane of order ρ, then $f'(z)$ is also of order ρ. *M. S. Robertson* (New Brunswick, N. J.).

Milloux, H. Sur une propriété des fonctions méromorphes et de leurs dérivées. J. Math. Pures Appl. (9) **31**, 1–18 (1952).

Using the formula of Jensen and the lemma of Schwarz, the author has shown elsewhere [J. Analyse Math. **1**, 244–330 (1951); these Rev. **13**, 930] that, if the modulus of a function, holomorphic in a circle, exceeds one, and if the derivative of the function vanishes a large number of times in an interior circle, the relative variation of the function in the interior circle is quite feeble. In this paper the author gives a new demonstration, extending the results to meromorphic functions, by making use of the second fundamental inequality of R. Nevanlinna. If $g(x)$ is meromorphic in the unit circle and does not take more than n times the three values a, b, c, whose spherical distances two at a time exceed ϵ^{-n}, if the derivative of $g(x)$ vanishes n' times in an interior circle $|x|\leqq\lambda<1$, and if n'/n is sufficiently large, then in almost all of the interior circle the variation of the point representing the function on the Riemann sphere is small. When the points a, b, c are 0, 1, ∞, the result obtained is quantitatively more precise. *M. S. Robertson.*

Ozaki, Shigeo. On the theory of multivalent functions. II. Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. **4**, 45–87 (1941).

Let $f(z)$ be meromorphic in $|z|<r$, continuous (and $f'(z)\neq 0$) on $|z|=r$. If $n(r)$ denotes the order of multivalency of $f(z)$ and $n(r, a)$ the number of a-points in $|z|<r$, then for $z=re^{i\theta}$

$$\frac{1}{2\pi}\int_0^{2\pi}\left|1+\Re\frac{zf''(z)}{f'(z)}\right|d\theta\geqq\max\{1, n(r)-n(r, \infty)\}.$$

If also $-m\leqq 1+\Re(zf''(z)/f'(z))\leqq M$ $(M, m>0)$ for $|z|<r$, and if $n^*(r, a)$ denotes the number of a-points of $f'(z)$ in $|z|<r$, then

$$\frac{[n(r)-n(r, \infty)]+[n^*(r, 0)-n^*(r, \infty)+1]}{M}+\frac{[n(r)-n(r, \infty)]-[n^*(r, 0)-n^*(r, \infty)+1]}{m}<2.$$

Conversely, if there exists a relation

$$\frac{[k-n(r,\infty)]+[n^*(r,0)-n^*(r,\infty)]+2}{M}+\frac{[k-n(r,\infty)]-[n^*(r,0)-n^*(r,\infty)]}{m}\geqq 2,$$

then $f(z)$ is at most k-valent for $|z|<r$. From the above fundamental theorems numerous conditions are obtained, any one of which is sufficient for $f(z)$, meromorphic in $|z|<r$, to be k-valent in $|z|<r$.

Let ρ_k be the radius of k-valency of $f(z)$, meromorphic in $|z|<R\leqq\infty$. Let $b_1, b_2, \cdots, b_{n(\rho_k,\infty)}$ be the poles of $f(z)$ in $|z|<\rho_k$ arranged according to increasing moduli, multiplicity counted. Let

$$2\pi\rho_k M=\int_0^{\rho_k}\left|1+\Re\frac{zf''(z)}{f'(z)}\right|dr \quad (z=re^{i\theta}).$$

Then

$$\sum_{n=1}^{k-1}\rho_n-\sum_{n=1}^{n(\rho_k,\infty)}|b_n|\geqq(k-M-n(\rho_k,\infty))\rho_k.$$

Let $f(z)$ be regular for $|z|\leqq 1$, $f(0)=0$, $f'(0)=1$. There exists a diametral line $z_1 0 z_2$ of the unit circle $|z|=1$ such that $w_1=f(z_1)$, the origin and $w_2=f(z_2)$ lie on a straight line in that order. If $f(z)$ is regular and star-like in the direction $w_1 0 w_2$ of $f(z)$ for $|z|\leqq 1$, then

$$|a_{n+1}z_1^2-a_{n-1}|\leqq 2, \quad f(z_1)=w_1, \quad \text{and} \quad |a_n|\leqq n.$$

The reviewer remarks that these inequalities were also obtained independently about the same time by de Bruijn [see Nederl. Akad. Wetensch., Proc. **44**, 47–49 (1941); these Rev. **2**, 274]. For the odd typically-real functions

$$f(z)=z+a_3z^3+a_5z^5+\cdots, \quad |z|\leqq 1,$$

which are star-like in the direction of the imaginary axis, $|a_{n-1}|+|a_{n+1}|\leqq 2$, $n\geqq 2$. This result is not new [see M. S. Robertson, Bull. Amer. Math. Soc. **41**, 565–572 (1935)] but follows easily as a corollary to the author's results.

M. S. Robertson (New Brunswick, N. J.).

Ozaki, Shigeo. On the theory of multivalent functions in a multiply connected domain. Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. **4**, 115–135 (1944).

If both $f(z)$ and $\sum_0^{p-1}c_nz^n+f(z)$ are at most p-valent in a domain $\mathfrak{D}$ for arbitrary $c_0, c_1, \cdots, c_{p-1}$, $f(z)$ is said to be absolutely p-valent in $\mathfrak{D}$. Six conditions, any one of which is sufficient for $f(z)$ to be regular and absolutely p-valent in $r<|z|<R$, are obtained when $f(z)$ is of the form $g(z)+h(z)$, where $g(z)=\sum_1^\infty a_{-n}z^{-n}$, $h(z)=\sum_0^\infty a_nz^n$. One such condition is

$$|a_p|\geqq\sum_{p+1}^{\infty}\binom{n}{p}|a_n|R^{n-p}+\sum_1^{\infty}\left|\binom{-n}{p}a_{-n}\right|r^{-n-p}.$$

The result is extended to the case where $h(z)$ is regular in a convex domain $\mathfrak{D}$ and $g(z)$ is regular in a reciprocally convex domain E.

Let $f(z)$ be meromorphic in a multiply connected domain $\mathfrak{D}$, not containing the point at infinity, and bounded by two closed convex curves c_1 and c_2. Let $n(\mathfrak{D}, a)$ denote the number of a-points of $f(z)$ in $\mathfrak{D}$, and define $n_k(\mathfrak{D},0)-n_k(\mathfrak{D},\infty)$ by the expression

$$2\pi(n_k(\mathfrak{D},0)-n_k(\mathfrak{D},\infty))=\int_{c_1}\frac{d\arg d^k f(z)}{d\arg dz}d\phi-\int_{c_2}\frac{d\arg d^k f(z)}{d\arg dz}d\phi,$$

where

$$-m<\frac{d\arg d^k f(z)}{d\arg dz}<M \text{ on } C_1,$$

$$-M<\frac{d\arg d^k f(z)}{d\arg dz}<m \text{ on } C_2 \quad (M, m\geqq 0).$$

If

$$\{[p+1-n(\mathfrak{D},\infty)]+[n_k(\mathfrak{D},0)-n_k(\mathfrak{D},\infty)]\}/M+\{[p+1-n(\mathfrak{D},\infty)]-[n_k(\mathfrak{D},0)-n_k(\mathfrak{D},\infty)]\}/m\geqq 4,$$

then $f(z)$ is at most p-valent in $\mathfrak{D}$. Numerous consequences of this fundamental theorem are obtained for meromorphic functions to be either p-valent or absolutely p-valent in $r<|z|<R$. *M. S. Robertson* (New Brunswick, N. J.).

Bernardi, S. D. A survey of the development of the theory of schlicht functions. Duke Math. J. **19**, 263–287 (1952).

This article surveys the history and some of the recent developments of schlicht functions, and especially the class S of functions $f(z)=z+a_2z^2+a_3z^3+\cdots$ which are regular and schlicht in $|z|<1$. Two types of theorems which are discussed may be described: (i) those of the same depth as the inequality $|a_2|\leqq 2$, (ii) those which refer to a subclass of S in which the question may be referred to functions with positive real part. Other topics treated are the bounds for $|f(z)|$, $|f'(z)|$, $|\arg f'(z)|$, covering theorems, variational methods, and coefficient domains. A number of still open problems are discussed and some questions raised for future work. *A. C. Schaeffer* (Madison, Wis.).

Wolibner, W. Sur certaines conditions nécessaires et suffisantes pour qu'une fonction analytique soit univalente. Colloquium Math. **2** (1951), 249–253 (1952).

Let $f(z)=z+\sum_{k=1}^{\infty}b_kz^{-k}$ be regular in $1<|z|<\infty$, and let $W_n[f(z)]=\sum_{k=-n}^{\infty}c_kz^{-k}$ where $W_n(u)$ is a polynomial of degree n. The author proves that a sufficient condition for the univalence of $f(z)$ in $|z|>1$, is that $\sum_{k=-n}^{\infty}k|c_k|^2\leqq 0$ for every polynomial W. The necessity of this condition was proved earlier by Goluzin [Mat. Sbornik **8(50)**, 277–284 (1940); these Rev. **2**, 185] and Biernacki [Bull. Sci. Math. **70**, 51–76 (1946); these Rev. **8**, 326]. *A. W. Goodman.*

Kolbina, L. I. On the theory of univalent functions. Doklady Akad. Nauk SSSR (N.S.) **84**, 1127–1130 (1952). (Russian)

For the class of functions $F(\zeta)=\zeta+\alpha_0+\alpha_1/\zeta+\cdots$ regular and univalent in $1<|\zeta|<\infty$, the author obtains an inequality which is too complicated to reproduce here. This inequality contains as special cases a variety of inequalities obtained earlier by Golusin, Lebedev, Schiffer, and Nehari.

A. W. Goodman (Lexington, Ky.).

Kolbina, L. I. Some extremal problems in conformal mapping. Doklady Akad. Nauk SSSR (N.S.) **84**, 865–868 (1952). (Russian)

Let $f_1(\zeta)$, $f_2(\zeta)$ be two functions each regular and univalent in $|\zeta|<1$, and suppose the two image regions have no points in common. Using variational methods developed by Goluzin [Mat. Sbornik N.S. **19(61)**, 203–236 (1946); these Rev. **8**, 325], the author sketches a proof that for $\alpha, \beta>0$, $\alpha\neq\beta$,

$$|f_1'(0)|^\alpha|f_2'(0)|^\beta\leqq\frac{4^{\alpha+\beta}\alpha^\alpha\beta^\beta}{|\alpha-\beta|^{\alpha+\beta}}\left|\frac{\sqrt{\alpha}-\sqrt{\beta}}{\sqrt{\alpha}+\sqrt{\beta}}\right|^{2\sqrt{(\alpha\beta)}}|a_1-a_2|^{\alpha+\beta}$$

where $f_k(0)=a_k$ $(k=1,2)$. A similar result involving three functions is announced. *A. W. Goodman.*

Kubo, Tadao. On conformal mapping of multiply-connected domains. Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **26**, 211–223 (1951).

It is proved that a domain of finite connectivity in the complex plane can be mapped conformally onto a parallel strip furnished with parallel slits. In the case of a circular ring, an explicit representation of the mapping function in question is given. *Z. Nehari* (St. Louis, Mo.).

Lelong-Ferrand, Jacqueline. Sur certaines classes de représentations d'un domaine plan variable. J. Math. Pures Appl. (9) **31**, 103–126, 245–252 (1952).

The author defines the kernel $\Omega(m_0, D_n)$ of a sequence of plane domains D_n with respect to a point m_0 to be the component of $\liminf D_n$ which contains m_0. She considers a sequence of topological mappings T_n of D_n onto domains Δ_n which have bounded Dirichlet integrals. She gives further conditions on T_n, D_n, $\tau_n = T_n^{-1}$, and Δ_n which guarantee that from each subsequence of the T_n, a subsequence T_{q_n} can be selected which converges to a topological mapping T of $\Omega(m_0, D_{q_n})$ onto $\Omega(\mu_0, \Delta_{q_n})$, and τ_{q_n} converges to $\tau = T^{-1}$. This theorem is a generalization of part of the theorem of Carathéodory [Math. Ann. **72**, 107–144 (1912)] on the conformal mapping of variable domains, the condition of conformality replaced here by much weaker conditions. A study is also made of degenerate cases not considered by Carathéodory in which the limit mapping takes $\Omega(m_0, D_{q_n})$ into a single point μ_0, etc. The author includes a discussion of uniqueness conditions for the limit mapping, and the correspondence of prime ends on the boundary. An error was made in setting up the pages for printing. In order to read it coherently, one skips from the end of line 1, p. 106 to line 2, p. 108; from the end of line 1, p. 109 to line 2, p. 107; from the end of line 1, p. 108 to line 2, p. 106; and from the end of line 1, p. 107 to line 2, p. 109. Pages 245–252 give a corrected version of pages 103–110. *G. Springer.*

Wolontis, Vidar. Properties of conformal invariants. Amer. J. Math. **74**, 587–606 (1952).

Let D be a region of the complex plane, E_1 and E_2 two disjoint compact subsets of D, and Γ the set of all rectifiable curves γ in D joining E_1 and E_2. Let R be the set of all functions $\rho \geqq 0$ such that $L(\rho; \gamma) = \int_\gamma \rho |dz|$ exists; we assume also that $A(\rho; D) = \iint_D \rho^2 dx dy$ is finite and $\neq 0$. We define by

$$\sup_{\rho \subset R} \inf_{\gamma \subset \Gamma} [L(\rho; \gamma)]^2 / A(\rho; D)$$

the extremal distance $\lambda_D(E_1, E_2)$ [cf. Ahlfors and Beurling, Acta Math. **83**, 101–129 (1950); these Rev. **12**, 171]. In the present paper the following results concerning this quantity are proved.

1. The extremal distance is a continuous set-function of the sets E_1 and E_2.

2. Let E_1 and E_2 be bounded by a finite number of analytic arcs. We denote by μ_i arbitrary Borel unit distributions on the boundary E_i' of E_i. We define by $G(\zeta, z_1, z_2)$, $z_1 \neq z_2$, the "Green function" of D characterized by the following conditions: (a) $G(\zeta, z_1, z_2) - \log |(\zeta - z_2)/(\zeta - z_1)|$ is harmonic for $\zeta \subset D$;

(b) $$G(\zeta, z_1, z_2) = G(\bar{\zeta}, z_1, z_2),$$
$$\partial G(\zeta, z_1, z_2)/\partial n = -\partial G(\bar{\zeta}, z_1, z_2)/\partial n$$

provided both ζ and $\bar{\zeta}$ belong to the boundary of D; (c) $\partial G/\partial n = 0$ for all the other boundary points. We denote by

$$p(z_1, z_2) = \int_{E_1'} G d\mu_1(\zeta) - \int_{E_2'} G d\mu_2(\zeta)$$

and write $d = \min p(z_1, z_2)$ where $z_i \subset E_i'$. We have then

$$2\pi \lambda_D(E_1, E_2) = \max d \quad \text{(for all possible } \mu_i\text{)}.$$

3. Finally a proper definition of "circular symmetrization" is given and it is shown that $\lambda_D(E_1, E_2)$ is increased if D remains fixed and the E_i are subjected to this process. The special case of logarithmic capacity is treated. Concerning this case cf. Pólya and Szegö [Isoperimetric inequalities in mathematical physics, Princeton Univ. Press, 1951, pp. 182–216; these Rev. **13**, 270]. *G. Szegö.*

Ohtsuka, Makoto. On the behavior of an analytic function about an isolated boundary point. Nagoya Math. J. **4**, 103–108 (1952).

Soit D un domaine du plan z, z_0 un point frontière isolé de D, R une surface de Riemann abstraite, $f(z)$ une fonction analytique de D dans R, D_r l'ensemble des valeurs prises par f pour $0 < |z - z_0| < r$, D_r^* l'adherence de D_r, S et S^* les intersections des D_r et des D_r^* quand r varie. L'auteur démontre les résultats suivants. 1) Si S^* n'est pas vide, elle comprend ou un seul point ou tous les points de R. 2) Ou bien, quand $r \to 0$, D_r converge vers un point de R ou vers une composante frontière idéale parabolique de R; ou bien, S est conformément équivalente à une sphère avec deux points exceptionnels possibles, ou à un tore.

R. de Possel (Alger).

Ohtsuka, Makoto. On a covering surface over an abstract Riemann surface. Nagoya Math. J. **4**, 109–118 (1952).

Soit $\mathbf{R}$ une surface de Riemann abstraite, R un recouvrement ouvert étalé sur $\mathbf{R}$. Le but essentiel du mémoire est de définir sur R une fonction $\omega(P)$ dite mesure harmonique de la frontière accessible de R par rapport à $\mathbf{R}$, et de montrer que si $\omega(P) > 0$, il existe sur R une fonction harmonique bornée non constante. L'auteur a déjà traité des questions voisines [même J. **3**, 91–137 (1951); ces Rev. **13**, 642]. Voici la définition de $\omega(P)$. Soient C les chemins sur R qui tendent vers un point de la frontière idéale de R, mais dont la projection sur $\mathbf{R}$ tend vers un point de $\mathbf{R}$; $v(P)$ les fonctions surharmoniques continues sur R, telles que $0 \leqq v \leqq 1$, $\lim v(P) = 1$ le long des chemins C; $\mu(P, R)$ l'enveloppe inférieure des $v(P)$, qui est harmonique d'après le principe de Perron-Brelot. On désigne par $\mathbf{R}'$ la projection de R sur $\mathbf{R}$, et par l'indice ∞ le recouvrement universel d'une surface. Cas 1: Si $(\mathbf{R}')^*$ est hyperbolique, on pose $\omega(P) = \mu(P, R^*)$. Cas 2: Sinon, on choisit 1, 2, ou 3 points sur $\mathbf{R}$, et on retire de R les points qui ont ceux-là pour projections, de sorte que la surface obtenue $\bar{R}$ ait un recouvrement universel hyperbolique; on pose $\omega(P) = \mu(P, \bar{R}^*)$, et on montre que la définition est indépendante du choix.

Indiquons quelques autre résultats. Représentons conformément R^* sur le disque U: $|z| < 1$. Les images des chemins C tendent vers les points d'un ensemble E sur T: $|z| = 1$. (1) Dans le premier cas, E est mesurable sur T, et la mesure harmonique de E en z est égale à $\omega(P)$ au point P qui correspond à z. (2) Dans les deux cas, si R est du type F, c'est-à-dire, si la fonction qui représente U dans $\mathbf{R}$ a une limite presque partout sur T le long de tout chemin non tangent, on a $\omega(P) = 1$. (3) Dans le deuxième cas, si R a une frontière positive, on a $\mu(P, R) = \mu(P, \bar{R}) \geqq \mu(P, R^*) \geqq \omega(P)$, mais on n'a pas d'exemple où l'une des deux inégalités serait strictement vérifiée. Si R a une frontière nulle, on a $\mu(P, R) = \mu(P, \bar{R}) = 1$, $\mu(P, R^*) = 0$. Enfin, R est dite du type D si toute fonction sous-harmonique $u(P)$ continue, bornée supérieurement, telle que $\limsup u(P) \leqq 0$ le long des

chemins C, est partout $\leqq 0$. Pour que R soit du type D, il faut et il suffit que $\mu(P, R)=1$, et alors $\omega(P)=1$.

R. de Possel (Alger).

Hitotumatu, Sin. Cousin problems for ideals and the domain of regularity. II. Proc. Japan Acad. **28**, 25–28 (1952).

Rappel de résultats connus et énoncé de deux théorèmes triviaux. *H. Cartan* (Paris).

Bureau, Florent. Sur les transformations engendrées par des systèmes de fonctions analytiques de plusieurs variables complexes. J. Math. Pures Appl. (9) **31**, 161–190 (1952).

Let n functions $f_j(z_1, \cdots, z_n)$, $j=1, \cdots, n$, be analytic in the unit hypersphere H: $\sum_1^n |z_j|^2 \leqq 1$, and zero at the origin. If $\sum_1^n |f_j(z)|^2 < 1$ in H, then Schwarz' lemma states that $\sum_1^n |f_j(z)|^2 \leqq \sum_1^n |z_j|^2$. The author shows that if the equality holds in the latter relation for all points of a hypersphere $\sum |z_j|^2 = \rho$ (ρ fixed, $0<\rho<1$), then it holds throughout, and the transformation $Z_j = f_j(z)$ is linear and unitary. Using this result he obtains the following theorem for two complex variables. Let $f(x, y)$, $g(x, y)$ be analytic in the unit hypersphere H': $|x|^2+|y|^2 \leqq 1$ and let $|f|^2+|g|^2 \leqq 1$ there. If (x_0, y_0) is any point in the interior of H', then

$$|f-f_0|^2+|g-g_0|^2 \leqq \frac{1+R_0}{1-R_0}\,\frac{|x-x_0|^2+|y-y_0|^2}{(1-rr_0)^2}$$

where

$$f_0=f(x_0, y_0),\quad g_0=g(x_0, y_0),\quad R_0^2=|f_0|^2+|g_0|^2,$$
$$r^2=|x|^2+|y|^2 \text{ and } r_0^2=|x_0|^2+|y_0|^2.$$

As a special case, if $(x_0, y_0)=(0, 0)$, this relation shows, in analogy to Schottky's theorem, that every analytic bounded transformation $(x, y)\to(f, g)$ in H' is such that $|f|^2+|g|^2$ remains bounded in every hypersphere $|x|^2+|y|^2 \leqq \theta < 1$ by a fixed number which depends only on (f_0, g_0) and θ. He obtains a related result on the Jacobian of f, g in analogy to Landau's theorem. In the second chapter the author applies the properties of Picard's hyperfuchsian functions to the study of transformations defined by pairs of analytic functions of two complex variables, and in the third chapter he uses these results in the study of normal families of analytic transformations. In particular, he obtains the following result. Let T be an analytic transformation defined by n functions $Z_j = f_j(z_1, \cdots, z_n)$, $j=1, \cdots, n$, each analytic in a bounded domain D of E_{2n}. A (not necessarily denumerable) family of transformations $\{T(\alpha)\}$ defined by functions analytic in D is normal in D if for each α the set of points $\{T(\alpha)\}$ has no points in common with the $n(n+3)/2$ hyperplanes $Z_j=0$, $Z_j=1$, $Z_j=Z_k$; $j, k=1, \cdots, n$, $j \neq k$.

W. T. Martin (Cambridge, Mass.).

Bellman, Richard. The iteration of power series in two variables. Duke Math. J. **19**, 339–347 (1952).

Pour une fonction $f(x)$ admettant l'origine pour point fixe attractif de multiplicateur a_1 non nul ($0 \leqq |a_1| < 1$), le problème de l'itération analytique a été résolu par Koenigs en 1884–1885: si $f_n(x)$ est l'itérée d'ordre n, la fonction de Koenigs $\phi(x)=\lim_{n\to\infty} f_n(x)/a_1^n$ vérifie l'équation $\phi(f(x))=a_1\phi(x)$ et l'itérée d'ordre t est alors $\phi^{-1}[a_1^t\phi(x)]$. Le cas des fonctions de plusieurs variables avait été abordé par Leau, Grévy et Lattès (non cité par l'auteur). Récemment, Hadamard [Bull. Amer. Math. Soc. **50**, 67–75 (1944); ces Rev. **5**, 185] et Töpfer [Math. Ann. **121**, 191–222 (1949); ces Rev. **11**, 168] sont revenus sur ces questions. L'auteur démontre que, moyennant certaines hypothèses, on peut généraliser le résultat de Koenigs; il traite explicitement le cas d'un couple de fonctions mis sous forme canonique

$$f(x, y)=\rho x+\sum_{h+l\geqq 2} a_{hl}x^h y^l,\quad g(x, y)=\sigma y+\sum_{h+l\geqq 2} b_{hl}x^h y^l \qquad (k, l \geqq 0),$$

les séries étant convergentes pour $|x|$ et $|y|$ assez petits. Si $0<|\rho|$, $|\sigma|<1$, $\rho \neq \sigma^k$, $k=2, 3, \cdots$, $\sigma \neq \rho^l$, $l=2, 3, \cdots$, il définit deux fonctions

$$\phi(x, y)=\sum_{h+l\geqq 1} c_{hl}x^h y^l,\quad \psi(x, y)=\sum_{h+l\geqq 1} d_{hl}x^h y^l \quad (k, l \geqq 0)$$

telles que $\phi(f, g)=\rho\phi(x, y)$, $\psi(f, g)=\sigma\psi(x, y)$. On a

$$\phi(x, y)=F(1/\rho, x, y),\quad \psi(x, y)=G(1/\rho, x, y);$$

les fonctions $F(z, x, y)$, $G(z, x, y)$ sont des fonctions méromorphes de z définies par prolongement analytique à partir d'expressions explicites valables dans un domaine restreint. La démonstration fait appel au théorème d'Hadamard sur la multiplication des singularités. *G. Valiron* (Paris).

Roussel, André. Une généralisation des développements de Taylor. Acta Math. **87**, 147–173 (1952).

The classical Taylor series development is generalized as follows: Let $g(x, h)$ be a function of two complex variables x and h, holomorphic for $|x-a|<R$, $|h|<R'$, and suppose $g(x, 0)=0$. The sequence $\{g_n(x, h)\}$ is formed by taking

$$g_1(x, h)=\int_0^h \frac{\partial g}{\partial x}dh, \cdots, g_n(x, h)=\int_0^h \frac{\partial g_{n-1}}{\partial x}dh, \cdots.$$

A second sequence $\{g_{-m}(x, h)\}$ is constructed by taking

$$g_{-1}(x, h)=\int_a^x \left\{\frac{\partial g}{\partial h}-\left(\frac{\partial g}{\partial h}\right)_{h=0}\right\}dx, \cdots, g_{-m}(x, y)$$
$$=\int_a^x \left\{\frac{\partial g_{-m+1}}{\partial h}-\left(\frac{\partial g_{-m+1}}{\partial h}\right)_{h=0}\right\}dx, \cdots.$$

Then, for $|x-a|+|h|<\min(R, R')$, the series

$$\sum_{m=1}^{\infty} g_{-m}(x, h)+g(x, h)+\sum_{n=1}^{\infty} g_n(x, h)$$

converges absolutely to the sum $f(x+h)-f(x)$, where

$$f(x)=g(a, x-a)+\int_0^{x-a} g_x'(x-t, t)dt.$$

The series reduces to the classical Taylor series on taking $g(x, h)=h\phi'(x)$, where $\phi(x)$ is holomorphic in $|x-a|<R$. The results obtained may be generalized to functions of several complex variables and the author illustrates how this is done for a function $g(x, y, h, k)$ of four complex variables [see C. R. Acad. Sci. Paris **225**, 23–24, 348–349 (1947); **227**, 500–502 (1948); these Rev. **9**, 21; **10**, 110].

M. S. Robertson (New Brunswick, N. J.).

Chow, Wei-Liang, and Kodaira, Kunihiko. On analytic surfaces with two independent meromorphic functions. Proc. Nat. Acad. Sci. U. S. A. **38**, 319–325 (1952).

Let V be a compact complex analytic manifold of complex dimension r, and let $\mathfrak{F}(V)$ denote the field of all meromorphic functions of V. Then, according to recent results of Chow, $\mathfrak{F}(V)$ is a finite algebraic function field whose degree of transcendency s is at most equal to r, and there exists an algebraic variety V^* of dimension s in a projective space, uniquely determined up to a birational transformation, and a meromorphic transformation Φ of V into V^* such that

every meromorphic function on V is the image of a meromorphic function on V^* under the inverse transformation Φ^{-1}. Thus Φ induces an isomorphism of the two fields $\mathfrak{F}(V)$ and $\mathfrak{F}(V^*)$. The variety V^* is called the algebraic equivalent of V, and V is said to be algebraic if V^* can be so chosen that Φ is bi-regular between V and V^*. The question arises: If V has r algebraically independent meromorphic functions, is it algebraic? The authors answer this question in the case in which V is a Kähler surface by proving the following theorem: Any Kähler surface with two algebraically independent meromorphic functions is an algebraic surface without singularities.

The proof makes use of quadratic transformations and the Riemann-Roch Theorem for Kähler surfaces recently established by Kodaira [Amer. J. Math. 73, 813–875 (1951); these Rev. 13, 981]. Let V be an analytic variety (not necessarily compact) of dimension 2, and let p be an arbitrary point of V. Then $\bar{V}$ is called the quadratic transformation of V (with center p) if there exists a holomorphic mapping P: $\bar{V}\to V$ such that $P^{-1}(p)$ is a complex projective line S. The transformation $\bar{V}=Q_p(V)=P^{-1}(V)$ is uniquely determined by V and p, and is the quadratic Cremona transformation of algebraic geometry applied to the surface V. Important applications of this transformation, of a different sort, have recently been made by H. Hopf and others.

If V is a Kähler surface, it is shown that $Q_p(V)$ is also a Kähler surface. Moreover, $Q_p(V)$ is algebraic if V is algebraic and conversely. The proof that $Q_p(V)$ is algebraic if V is follows well-known lines, while for the proof of the converse it is sufficient to show that the topological self-intersection number $I(S, S)$ is equal to -1, a fact which is easily established.

Now let V be a Kähler surface with two algebraically independent functions, in which case the algebraic equivalent V^* of V is of dimension 2 and may be assumed to be a non-singular algebraic surface in a projective space of dimension d. Let $(X_0, X_1, \cdots, X_d)$ be the homogeneous coordinates of a generic point $X\varepsilon V^*$, and denote by C_λ the inverse image $\Phi^{-1}(C_\lambda^*)$ of the hyperplane section C_λ^* of V^* cut out by the hyperplane $\lambda_0X_0+\cdots+\lambda_dX_d=0$. The mapping $p\to\Phi(p)$, $p\varepsilon V$, is regular except for a finite number of base points of the linear system $|C_\lambda|$. By applying a finite number of quadratic transformations, a new analytic surface $\bar{V}$ is obtained from V and the mapping $\bar{\Phi}$: $\bar{V}\to V^*$ is regular. Except for finitely many points, $\bar{\Phi}^{-1}(p^*)$ consists of a finite number s of points on $\bar{V}$ where s is independent of $p^*\varepsilon V^*$. By using the Riemann-Roch theorem for Kähler surfaces, it is shown that $s=1$ and hence that $\bar{\Phi}^{-1}$ is regular and single-valued except for a finite number of points on V^* each of which corresponds to a fundamental curve on $\bar{V}$. By applying a finite number of quadratic transformations to V^*, a new algebraic surface $\bar{V}^*$ is obtained such that the mapping $\bar{\Phi}_1$: $\bar{V}\to\bar{V}^*$ is bi-regular. Thus $\bar{V}$, and hence V, is algebraic.

D. C. Spencer (Princeton, N. J.).

Schmidt, Klaus. Über die Existenzgebiete regulärer Quaternionenfunktionen. Schr. Math. Inst. Univ. Münster, no. 4, iv+41 pp. (1951).

Im ersten Teil gibt Verf. eine kurze Einführung in die von Fueter begründete Theorie der regulären Funktionen einer Quaternionenvariablen und beweist dann den Satz, dass es zu jedem Gebiet G des hyperkomplexen Raumes R_4 eine reguläre Quaternionenfunktion gibt, welche genau G zu ihrem Existenzgebiet hat. Ziel des Hauptteiles der Arbeit ist die Untersuchung regulärer Funktionen zweier Quaternionenvariablen, insbesondere zu zeigen, dass jener Satz über Existenzbereiche nicht mehr gilt, sondern dass hier ähnliche Verhältnisse vorliegen, wie sie über analytische Funktionen zweier komplexen Veränderlichen bekannt sind, deren Existenzbereiche (Regularitätsbereiche) sich durch besondere Eigenschaften auszeichnen. So werden der Kontinuitätssatz und seine unmittelbaren Folgerungen und der Satz über die gleichzeitige Fortsetzbarkeit bewiesen, ebenso die Regulärkonvexität als notwendige Eigenschaft der Regularitätsbereiche nachgewiesen, wobei es noch offen bleibt, ob diese auch eine hinreichende ist. Schliesslich werden als einfachste Beispiele die planarkonvexen Bereiche—die stets Regularitätsbereiche sind—und die Rheinhardtschen Körper untersucht.

Zur Definition der regulären Funktionen von Quaternionenvariablen sei noch bemerkt, dass die Nichtkommutativität des Quaternionenkörpers zur Unterscheidung der links- und rechtsregulären Funktionen einer Quaternionenvariablen führt, die jedoch die gleichen wesentlichen Eigenschaften aufweisen. Beim Übergang zu zwei Quaternionenvariablen zeigt es sich aber, dass nur die in einer der beiden Variablen links- und in der andern rechtsregulären Funktionen sich auf einfache Weise durch Reihenentwicklungen nach regulären Elementarfunktionen charakterisieren lassen, und nur solche Funktionen werden hier betrachtet. Verf. weist darauf hin, dass reguläre Funktionen von mehr als zwei Quaternionenvariablen eine einfache Charakterisierung durch derartige Reihenentwicklungen überhaupt nicht mehr zulassen. Es ist also eine Verallgemeinerung der Theorie auf dem eingeschlagenen Wege über zwei Quaternionenvariablen hinaus nicht möglich.

P. Thullen (Genf).

Ozaki, Shigeo, Ono, Isao, and Ozawa, Mitsuru. On the function-theoretic identities in the three dimensional space. Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. 4, 195–202 (1951).

Just as the complex plane may be mapped onto the surface of a three-dimensional sphere we may set up a one-to-one, continuous correspondence between the whole 3-dimensional space $S(x_1, x_2, x_3)$ and the surface of a unit hypersphere in the 4-dimensional space $S_4(X_1, X_2, X_3, X_4)$, the point at infinity corresponding to (0, 0, 0, 1). A study is made here of some properties of quasi-meromorphic mappings, in particular the volume theorem and the principle of solid angle, corresponding to the area theorem and the principle of argument in the case of the complex plane. The methods used are similar to those of the author's earlier paper [see same Rep. 4, 157–160 (1951); these Rev. 13, 453].

M. S. Robertson (New Brunswick, N. J.).

Kuz'mina, A. L. On a class of quasi-analytic functions of several variables. Doklady Akad. Nauk SSSR (N.S.) 80, 853–856 (1951). (Russian)

Comme le remarque l'auteur lui-même, des résultats analogues aux siens ont été obtenus par Lelong [C. R. Acad. Sci. Paris 232, 1178–1180 (1951); ces Rev. 13, 26].

S. Mandelbrojt (Houston, Tex.).

Stiefel, E. On Cauchy-Riemann equations in higher dimensions. J. Research Nat. Bur. Standards 48, 395–398 (1952).

The n linear partial differential equations with constant complex coefficients,

$$(*)\qquad l_j=\sum_{i,k=1}^{n} a_{jk}^{i}\frac{\partial\mu_k}{\partial x_i}=0 \quad (j=1,\cdots,n),$$

are said to form a system of generalized Cauchy-Riemann equations provided there exist constants b^i_{jk} such that

$$(**) \qquad \Delta u_j \equiv \sum_{i,k=1}^{n} b^i_{jk} \frac{\partial l_k}{\partial x_i},$$

where Δu_j denotes the Laplacian of u_j. Clearly any set of solutions must be a set of harmonic functions. The necessary restriction on n previously given by Olga Taussky [Quart. J. Math., Oxford Ser. 10, 99–103 (1939); these Rev. 1, 15] is now improved by showing that [with trivial exceptions] there are systems for which (*) and (**) are consistent only in the cases $n=1, 2, 4, 8$. Further, the systems are given explicitly: in the cases $n=2, 4$ there are three essentially different systems, while for $n=8$ there are only two. If the coefficients are required to be real, there are only the classic system of two equations, the two Dirac-Fueter systems of four equations, and two systems of eight equations.

E. F. Beckenbach (Los Angeles, Calif.).

Masani, P., and Vijayaraghavan, T. An analogue of Laurent's theorem for a simply connected region. J. Indian Math. Soc. (N.S.) 16, 25–30 (1952).

Let B be a complex Banach algebra with a unit element. Let D_1, D_2 be open disks in the complex plane, with $D=D_1 \cap D_2$ nonvoid. Let f be a function with values in B, defined, analytic, and having an inverse $\{f(z)\}^{-1}$ at each point of D. The authors' theorem is that there exist functions f_1, f_2 such that $f(z)=f_1(z)f_2(z)$ on D, with f_i defined, analytic, and having an inverse on D_i $(i=1, 2)$. This is a modification of a special case of a theorem of H. Cartan [J. Math. Pures Appl. 19, 1–26 (1940); these Rev. 1, 312]. From Cartan one knows that if f is assumed analytic and possessed of an inverse on a neighborhood of $\bar{D}$, then there exist functions f_1, f_2 such that $f(z)=f_1(z)f_2(z)$, where f_i is analytic and has an inverse on a neighborhood of D_i. The transition from Cartan's theorem to the theorem of the present paper is made by the construction of convergent sequences of functions, using sequences of concentric circles expanding toward D_1 and D_2 from within.

A. E. Taylor.

Theory of Series

Chow, Hung Ching. A note on summable series. J. London Math. Soc. 27, 352–355 (1952).

A theorem of Knopp and Hardy, showing that a series $\sum a_n$ is evaluable to s by the Cesàro method $(C, 1)$ if and only if $s_n+(n+1)b_{n+1} \to s$ where $s_n=a_0+a_1+\cdots+a_n$ and $b_n=a_n/(n+1)+a_{n+1}/(n+2)+\cdots$ was generalized by Ferrar [same J. 1, 175–179 (1926)] to cover (C, k) when k is a positive integer. The present paper shows that the theorem of Ferrar remains valid when k is positive and not an integer.

R. P. Agnew (Ithaca, N. Y.).

Garreau, G. A. Methods of generating T-matrices. Nederl. Akad. Wetensch. Proc. Ser. A. 55 = Indagationes Math. 14, 237–244 (1952).

When a sequence σ_n of positive constants and a sequence $f_n(x)$ of functions satisfy stated conditions, the matrix a_{nk} defined by

$$(*) \qquad a_{nk}=\sigma_n^{-1}\int_k^{k+1} f_n(u/\sigma_n)du$$

is a regular matrix for evaluation of sequences. Moreover each regular matrix is representable in the form (*). Examples and related results on series-to-sequence transformations are given.

R. P. Agnew (Ithaca, N. Y.).

Agnew, Ralph Palmer. Equivalence of methods for evaluation of sequences. Proc. Amer. Math. Soc. 3, 550–556 (1952).

The following simple and useful Mercerian theorem is proved: (M) If the triangular method of summation $A=(a_{nk})$ is regular and if $\liminf_{n\to\infty}\{|a_{nn}|-\sum_{k<n}|a_{nk}|\}>0$, then A is equivalent to convergence. The author remarks that the method of proof given was first used by him [Tôhoku Math. J. 35, 244–252 (1932)] to obtain some slightly weaker variants of (M). He gives several illustrative examples. [It should be added that (M) was also given by Rado, Quart. J. Math., Oxford Ser. 9, 274–282 (1938)].

G. G. Lorentz.

de Castro Brzezicki, A. On continuable and noncontinuable Dirichlet series. Memorias de Matemática del Instituto "Jorge Juan," no. 13, 63 pp. (1951). (Spanish)

This monograph is a compilation of several previously published papers of the author [cf. Revista Acad. Ci. Madrid 43, 139–166, 359–391 (1949); these Rev. 11, 649; 13, 122].

R. C. Buck (Madison, Wis.).

Wintner, Aurel. On the non-vanishing of certain Dirichlet series. Rend. Circ. Mat. Palermo (2) 1, 35–39 (1952).

Let $f(mn)=f(m)f(n)$ for all pairs of positive integers m and n, let $f(1)\neq 0$, and write $F(s)=\sum_{n=1}^{\infty} f(n)n^{-s}$. Suppose that this series is absolutely convergent for $\sigma=\Re s>1$ and that the function $F(s)$, represented by the series for $\sigma>1$, remains regular on the line $\sigma=1$. Also suppose that $f(p)\geqq -1$ for all primes p. Then it is proved that (*) $F(1+it)\neq 0$ for every $t\neq 0$ by taking $\varphi(s)=-d\{\log \zeta(s)F(s)\}/ds$ in the following lemma: Suppose that the Dirichlet series $\varphi(s)=\sum_{n=1}^{\infty} c_n n^{-s}$ converges for $\sigma>1$, that $c_n\geqq 0$, and that the limit $g(t)=\lim_{\epsilon\to+0}\Re\varphi(1+\epsilon+it)$ exists for every t, where $-\infty<t<\infty$. Then (i) for every t, $-g(0)\leqq g(t)\leqq g(0)$, and (ii) if there exists some $t=t_0\neq 0$ for which $g(t_0)=-g(0)$, then there must exist some $t=t_1\neq 0$ for which $g(t_1)=g(0)$. This lemma is a straightforward abstraction from the classical proof of $\zeta(1+it)\neq 0$. In the proof of (*) no use is made of Landau's theorem on Dirichlet series with nonnegative coefficients. With the help of that theorem, however, the following result concerning the missing case $t=0$ is proved: If $F(s)$ is, in addition, regular in the half-plane $\sigma\geqq\frac{1}{2}$, then $F(1)\neq 0$.

R. A. Rankin (Birmingham).

Wintner, Aurel. On the non-vanishing of certain Dirichlet series. Amer. J. Math. 74, 723–725 (1952).

The following theorem is proved: Let $f(n)$ be a completely multiplicative and bounded function. Put $F(s)=\sum f(n)/n^s$ for $\sigma>1$. If $F(s)$ is free from singularities on $\sigma=1$, then it has at most one zero on $\sigma=1$. A somewhat more refined result was obtained by the author elsewhere if $f(n)$ is also real-valued [cf. the preceding review].

S. Agmon.

Mikusiński, J. G.-. On generalized power-series. Studia Math. 12, 181–190 (1951).

The author is interested in determining a class of series

$$(*) \qquad 1-\alpha_1 x^{\beta_1}+\alpha_2 x^{\beta_2}-\alpha_3 x^{\beta_3}+\cdots$$

with $\alpha_n>0$, $0<\beta_n\uparrow\infty$ which converge for $x\geqq 0$ to a monotone continuous function tending to zero as $x\to\infty$. His most handy result is that if the exponents satisfy $\inf(\beta_{n+1}-\beta_n)>0$, $\beta_n=pn+O(1)$ with some fixed p, then (*) with

$$\alpha_n=\prod_{\nu\neq n}(\beta_\nu/|\beta_\nu-\beta_n|)e^{-\beta_n/\beta_\nu}$$

has the required properties. The method of proof is quite elementary. The author does not mention that many of his auxiliary results are well-known facts about general Dirichlet series [see, e.g., G. Valiron, Théorie générale des séries de Dirichlet, Gauthier-Villars, Paris, 1926]. *A. Dvoretzky.*

Mikusiński, J. G.-. A theorem on moments. Studia Math. **12**, 191–193 (1951).

Using the result stated in the preceding review, the author proves the following theorem: Let $(\beta_n)_{n=1}^{\infty}$ satisfy the conditions imposed in the preceding review and $(\gamma_n)_{n=1}^{\infty}$ be any sequence of positive numbers tending to infinity. If $f(x)$ is integrable in the interval $0<a<x<b<\infty$ and if for any given $c>a$ there exists a finite number M such that $|\int_a^b x^{\gamma_m-\beta_n} f(x)dx| < Mc^{\gamma_m-\beta_n}$ for $m, n=1, 2, \cdots$, then $f(x)=0$ almost everywhere in (a, b). *A. Dvoretzky* (Jerusalem).

Leja, F. Remarques sur les séries entières doubles. Ann. Soc. Polon. Math. **24** (1951), 19–24 (1952).

If a diagonal series

$$\sum_{n=0}^{\infty} P_n(x, y), \quad P_n(x, y) = \sum_{m=0}^{n} a_{m,\,n-m} x^m y^{n-m},$$

is absolutely convergent on the half-circle $x=r\cos t$, $y=r\sin t$, $0\leqq t<\pi$, $r>0$, then it is also absolutely convergent for $|x+iy|<r$, $|x-iy|<r$, and the corresponding double series $\sum_{p,q} a_{p,q} x^p y^q$ is absolutely convergent for $|x|+|y|<r$. *S. Bochner* (Princeton, N. J.).

Whittaker, E. T. On the reversion of series. Gaz. Mat., Lisboa **12**, no. 50, 1 (1951).

Combining the Lagrange-Bürmann-Teixeira theorem and an explicit determinant expression for the Taylor coefficients of $[f(x)]^{-n}$ in terms of those of $f(x)$ due to H. W. Segar [Messenger of Math. **21**, 177–188 (1892)], the author obtains an explicit power series development of the inverse of an analytic function in the neighborhood of a simple zero. *A. Dvoretzky* (Jerusalem).

Thorkelsson, Thorkell. Serial relations. II. Soc. Sci. Islandica, Greinar **1**, 177–182 (1940).

Thorkelsson, Thorkell. A group of asymptotic series. Serial relations. III. Soc. Sci. Islandica, Greinar **1**, 189–193 (1940).

Thorkelsson, Thorkell. Differential series of Eulerian type. Serial relations. IV. Soc. Sci. Islandica, Greinar **1**, 201–208 (1940).

Thorkelsson, Thorkell. Asymptotic solutions of differential equations. Serial relations. V. Soc. Sci. Islandica, Rit **27**, 42 pp. (1946).

Thorkelsson, Thorkell. Serial relations and symbolic calculus. Soc. Sci. Islandica, Rit **29**, 124 pp. (1951).

A serial relation is a recurrent relation between coefficients of a power series, or a relation between coefficients of several power series. Such a relation may be expressed as a functional equation for the function or functions represented by the power series, and may then be presumed to hold even if the power series in question do not converge. Thus one arrives at an interpretation of divergent power series. For instance, if $y=\sum_0^{\infty} a_n x^n$, $z=\sum_0^{\infty} b_n x^n$, and $b_n=f(n)a_n$, then one has the "conversion formula"

$$z=\sum_{k=0}^{\infty} \frac{x^k}{k!} \frac{d^k y}{dx^k} \Delta_k f(0)$$

in which $\Delta_k f$ is the kth forward difference (for unit interval). The author uses this principle in the discussion of a variety of topics including some asymptotic expansions, differential equations, numerical computations, the operational calculus of differential and difference operators. He notes that some of these topics have been discussed by others from a different point of view. *A. Erdélyi* (Pasadena, Calif.).

Fourier Series and Generalizations, Integral Transforms

Timan, A. F. Linear methods of approximation of periodic functions by trigonometric polynomials. Doklady Akad. Nauk SSSR (N.S.) **84**, 1147–1150 (1952). (Russian)

Timan, A. F. On linear methods of approximation of functions. Uspehi Matem. Nauk (N.S.) **7**, no. 3(49), 103–112 (1952). (Russian)

Let $\lambda_k^{(n)}$ be any triangular matrix of numbers ($n=0, 1, 2, \cdots$; $0\leqq k\leqq n$) such that $\lambda_0^{(n)}=1$. It associates with every integrable and periodic $f(x)\sim \frac{1}{2}a_0+\sum_1^{\infty}(a_k\cos kx+b_k\sin kx)$ a sequence of trigonometric polynomials

$$U_n(f; x; \lambda)=\tfrac{1}{2}a_0+\sum_{k=1}^{n}\lambda_k^{(n)}(a_k\cos kx+b_k\sin kx).$$

For every class $\mathfrak{M}$ of functions f we may consider the constants

$$E_n(\mathfrak{M}; x; \lambda)=\sup_{f\in\mathfrak{M}} |f(x)-U_n(f; x; \lambda)|$$

and we may ask about the behavior of these constants for various classes $\mathfrak{M}$ and for $n\to\infty$. The author states without proof a number of theorems in connection with these problems. The statements are too long to be given here. For this type of problem see also B. Sz. Nagy [Acta Sci. Math. Szeged **12**, Pars B, 204–210 (1950); these Rev. **11**, 656] and the literature cited therein. *A. Zygmund* (Chicago, Ill.).

Bari, N. K. On primitive functions and trigonometric series converging almost everywhere. Doklady Akad. Nauk SSSR (N.S.) **84**, 1117–1118 (1952). (Russian)

Let $f(x)$ be any measurable and finite-valued function defined on the interval $(0, 2\pi)$. Lusin showed [Mat. Sbornik **28**, 266–294 (1912)] that there is a continuous function $F(x)$, $0\leqq x\leqq 2\pi$, such that $F'(x)=f(x)$ almost everywhere. Recently it was proved by Menshov [ibid. **9(51)**, 667–692 (1941); these Rev. **3**, 106] that there exists a trigonometric series having coefficients tending to 0 and converging to $f(x)$ almost everywhere (neither the function F nor the series are uniquely determined). In the present note the author states (with only a sketch of proof) the following result, which unifies the other two. Under the previous assumptions concerning $f(x)$, there exists a continuous function $F(x)$ such that (1) $F'(x)=f(x)$ almost everywhere in $(0, 2\pi)$; (2) the term-wise differentiated Fourier series of $F(x)$ converges to $f(x)$ almost everywhere in $(0, 2\pi)$. *A. Zygmund* (Chicago, Ill.).

Menšov, D. E. On limits of indeterminacy of Fourier series. Mat. Sbornik N.S. **30(72)**, 601–650 (1952). (Russian)

The very well-known result of Kolmogorov [Fund. Math. **4**, 324–328 (1923)] asserts that there is an integrable function $f(x)$ of period 2π such that the partial sums of the Fourier series of f are unbounded at almost every point; in particular, the Fourier series diverges almost everywhere.

By modifying Kolmogorov's construction, Marcinkiewicz [ibid. 27, 38–69 (1936)] obtained an $f(x)$ whose Fourier series diverges boundedly almost everywhere. In the present paper the author obtains the following more general result. 1) Let $\varphi(x)$, $-\pi\leqq x\leqq\pi$, be measurable and non-negative (it may take the value $+\infty$ in a set of positive measure). Then there exists an integrable function $f(x)$ such that the partial sums $S_n(x)$ of the Fourier series of f satisfy almost everywhere the relations

$$\liminf_{n\to\infty} S_n(x)=f(x)-\varphi(x),\quad \limsup_{n\to\infty} S_n(x)=f(x)+\varphi(x)$$

(to grasp the significance of this result, one should compare it with the following theorem: If one of the functions $\liminf S_n(x)$ and $\limsup S_n(x)$ is finite almost everywhere in a set E of positive measure, so is the other function and we have $f(x)=\frac{1}{2}\{\liminf S_n(x)+\limsup S_n(x)\}$ almost everywhere in E: if $\liminf S_n(x)=-\infty$ almost everywhere in E, then $\limsup S_n(x)=+\infty$ almost everywhere in E, and conversely [see Marcinkiewicz and Zygmund, ibid. 26, 1–43 (1936)]). 2) For general (non-Fourier) trigonometric series one has the following analogous result: Let $F(x)$ and $G(x)$, $-\pi\leqq x\leqq+\pi$, be measurable functions satisfying the inequality $G(x)\leqq F(x)$, and such that at almost every point either both functions are finite or we have simultaneously $G(x)=-\infty$, $F(x)=+\infty$; then there is a trigonometric series with coefficients tending to zero and with partial sums $S_n(x)$ satisfying almost everywhere the equations $\liminf S_n(x)=G(x)$, $\limsup S_n(x)=F(x)$. 3) Let $f(x)$, $-\pi\leqq x\leqq+\pi$, be an arbitrary integrable function and $P\subset(-\pi,+\pi)$ an arbitrary closed non-dense set. Let $\varphi(x)$, $-\pi\leqq x\leqq+\pi$, be an arbitrary non-negative and measurable function. Then it is possible to modify $f(x)$ outside the set P in such a way that the Fourier series of the resulting function $g(x)$ have almost everywhere the limits of indetermination $g(x)-\varphi(x)$ and $g(x)+\varphi(x)$. *A. Zygmund.*

Matsuyama, Noboru. Notes on Fourier analysis. XXXII. On the summability $(C, 1)$ of the Fourier series. Proc. Japan Acad. 26, no. 7, 5–9 (1950).

It is known that (1) the Fourier series of the even function $\varphi(t)$ is summable $(C, 1)$ at the origin if $\varphi(t)=o\{(\log(1/t))^{-1}\}$ $(C, 1)$, (2) the Fourier series is summable $(R, \log n, 1)$ if $\varphi(t)=o(1)$ $(C, 1)$, (3) the condition in (1) cannot be replaced by $\varphi(t)=o\{(\log(1/t))^{-\alpha}\}$ $(C, 1)$, $0<\alpha<1$, nor indeed by

$$(*)\qquad \int_{0+}^{t} u^{-1}\varphi(u)du=o\{(\log(1/t))^{-\alpha}\}.$$

[For (1) see F. T. Wang, J. London Math. Soc. 22, 40–47 (1947); these Rev. 9, 182; for (2) see A. Zygmund, Bull. Int. Acad. Polon. Sci. Cl. Sci. Math. Nat. Ser. A. Sci. Math. 1924, 243–249, or, for a more general result, F. T. Wang, Tôhoku Math. J. 40, 142–159, 274–292 (1935), and Duke Math. J. 15, 5–10 (1948); these Rev. 9, 580; for (3) see S. D. Sinvhal, Ganita 1, 27–30 (1950); these Rev. 13, 228.] The author's first two results are included in (1) and (2) respectively. His third result is a corollary of (3) [condition (*) with $s<\alpha<1$]. *L. S. Bosanquet* (London).

Sunouchi, Gen-ichirô, and Yano, Shigeki. Notes on Fourier analysis. XXVIII. On some maximal theorems for the strong summability of Fourier series. Math. Japonicae 2, 53–58 (1951).

A new proof of the inequality

$$\left\{\int_0^{2\pi} T_k{}^p(x)dx\right\}^{1/p}\leqq A_{k,p}\left\{\int_0^{2\pi}|f(x)|^p dx\right\}^{1/p}\quad (p>1),$$

where

$$T_k(x)=\sup_{1\leqq n<\infty}\left[\frac{1}{n}\sum_{\nu=1}^{n}|S_\nu(x)|^k\right]^{1/k}\quad (k>0)$$

and $S_n(x)$ is the nth partial sum of the Fourier series of $f(x)$ [Sunouchi, Proc. Imp. Acad. Tokyo 10, 420–423 (1943); these Rev. 7, 247]. The authors state that the case $k\leqq 2$ is easily deduced from a theorem of Zygmund [Fund. Math. 30, 170–196 (1938)]. They deduce a theorem of Marcinkiewicz [C. R. Acad. Sci. Paris 208, 782–784 (1939)].

L. S. Bosanquet (London).

Yano, Shigeki. Notes on Fourier analysis. XXIX. An extrapolation theorem. J. Math. Soc. Japan 3, 296–305 (1951).

For the theory of Fourier series and other parts of real analysis one often has a set of inequalities of the type

$$(1)\qquad \left\{\int_0^{2\pi}|g(x)|^p dx\right\}^{1/p}\leqq A_p\left(\int_0^{2\pi}|f(x)|^p dx\right)^{1/p}\quad (p>1)$$

and

$$(2)\qquad \int_0^{2\pi}|g(x)|dx\leqq A_k\int_0^{2\pi}|f(x)|\log^k(1+f^2(x))dx+B_k\quad (k>0),$$

where $g(x)=T(f(x))$ is a certain function associated with $f(x)$. Generalising an argument of Titchmarsh in the case of the conjugate function $g(x)=\bar{f}(x)$ [cf. Hardy and Littlewood, Acta. Math. 54, 81–116 (1930)] the author proves the following useful theorem that shows that, in a wide range of cases, (2) is a consequence of (1). Theorem: Let T be a transformation which transforms every $f(x)$, integrable over $\langle a, b\rangle$, into a measurable function $T(f)$ satisfying

$$(\mathrm{i})\qquad f=\sum_0^\infty f_\nu \text{ implies } |T(f)|\leqq\sum_0^\infty|T(f_\nu)|$$

and

$$(\mathrm{ii})\qquad |T(-f)|=|T(f)|.$$

If then (1) holds with $A_p\leqq A(p-1)^{-k}$, for every $p>1$, then (2) holds for this $k>0$. This theorem is illustrated on hand of several applications. Unfortunately, the paper is full of serious misprints, and the proof of the theorem, although sound in principle, is given with lack of precision in detail.

W. W. Rogosinski (Newcastle-upon-Tyne).

Zamansky, Marc. Sur les fonctions absoluement continues et les conjuguées d'une fonction sommable. Ann. Mat. Pura Appl. (4) 32, 157–177 (1951).

Let $y(x, n)$ be a function of a real variable x and positive integer n. We write $y=O'(1)$ if for almost every x the sequence $y(x, n)$ is bounded. The symbol $o'(1)$ is defined similarly. Let $\varphi(x)$ be an L-integrable function of period 2π, with $\int_0^{2\pi}\varphi dx=0$, and let $f(x)$ be a primitive of φ. Let φ^* and f^* denote the conjugate functions of φ and f respectively, and $\sigma_n(x, f)$ and $\sigma_n^*(x, f)$ the $(C, 1)$ means of the Fourier series of f and of the conjugate series. We shall also write $\Delta_2(f, x, t)=f(x+2t)+f(x-2t)-2f(x)$. The present paper is concerned with the "degree of approximation almost everywhere". The author shows that

$$(1)\qquad \sigma_n(x, f)-f(x)=(n\pi)^{-1}\int_{1/n}^{\infty} t^{-2}\Delta_2(f, x, t)dt+o'(1/n);$$

$$(2)\qquad f^*(x)-\sigma_n^*(x)=\varphi(x)/n+o'(1/n),$$

$$(3)\qquad \sigma_n''(f, x)=o'(n),\quad \sigma_n''^*(f, x)=o'(n);$$

(4) results analogous to (1) and (2) hold for a general class of methods of summation. (5) Let E be the set of measure 2π in which $\varphi=f'$, φ^* and f^* exist simultaneously, and let H be any set of positive measure. A necessary and sufficient condition that at every point x of the set $H'=H\cap E$ the function f^* have φ^* as a derivative, with respect to the set H', is that

$$\varphi(x)=\lim_{h\to 0}\frac{1}{2h}\int_0^{2h}\frac{f(x+h+t)+f(x+h-t)}{2t}dt$$

for $x+h$ remaining in H'. *A. Zygmund* (Chicago, Ill.).

Žak, I. E. On absolute convergence of double Fourier series. Soobščeniya Akad. Nauk Gruzin. SSR. **12**, 129–133 (1951). (Russian)

The main result of the paper is an extension to two variables of the following well-known theorem [see the reviewer's Trigonometrical series, Warsaw-Lvov, 1935, p. 143]: If $f(x)$ is of period 2π, of bounded variation, and of class Lip α, and if $f\sim\sum c_n e^{inx}$, then $\sum|n|^{\gamma/2}|c_n|<\infty$ for every $\gamma<\alpha$. The precise statement is as follows. Suppose that $f(x, y)$ is of period 2π both in x and y, and that

$$|f(x_2, y)-f(x_1, y)|\leqq K_1(y)|x_2-x_1|^{\alpha_1},$$
$$|f(x, y_2)-f(x, y_1)|\leqq K_2(x)|y_2-y_1|^{\beta_1},$$
$$|f(x_2, y_2)-f(x_1, y_2)-f(x_2, y_1)+f(x_1, y_1)|\leqq K|x_2-x_1|^{\alpha}|y_2-y_1|^{\beta},$$

where α, β, α_1, β_1 are positive numbers less than 1, K is a constant, and $K_1(y)$, $K_2(x)$ are integrable functions. Suppose also that the variation of f over $0\leqq x\leqq 2\pi$ is an integrable function of y, and conversely. Finally, suppose that f is of bounded variation in the sense of Carathéodory (by this the author means that if we split the fundamental square into rectangles, by means of lines parallel to the axes, then the sum of the moduli of the increments of f over the rectangles stays below a finite number). Then, if $f(x, y)\sim\sum c_{mn}e^{i(mx+ny)}$, we have

$$\sum|c_{mn}|(|m|+1)^{\delta/2}(|n|+1)^{\gamma/2}<\infty,$$

where $\delta=\min(\alpha, \alpha_1)$, $\gamma=\min(\beta, \beta_1)$. *A. Zygmund.*

Verblunsky, S. Sur une classe de séries exponentielles de Cauchy. Bull. Sci. Math. (2) **76**, 85–96 (1952).

Let $0=\mu_0<\mu_1<\mu_2<\cdots$ be the non-negative roots of the equation $z+h\tan\pi z=0$, $h>0$. One has $\mu_\nu=\nu-\frac{1}{2}+O(\nu^{-1})$. Let $f(x)$ be real and belong to $L(-\pi, \pi)$. The author investigates the non-harmonic series

$$(*)\qquad f(x)\sim\tfrac{1}{2}a_0+\sum_{\nu=1}^{\infty}(a_\nu\cos\mu_\nu x+b_\nu\sin\mu_\nu x)$$

where

$$a_\nu+ib_\nu=\frac{h^2+\mu_\nu^2}{h+\pi(h^2+\mu_\nu^2)}\int_{-\pi}^{\pi}f(t)e^{i\mu_\nu t}dt.$$

This "Fouriersche Reihe der Abkühlung" of $f(x)$ was previously studied by Fejes (not by Fejér, as the author consistently writes, although Fejér did propose the name) [Math. Naturwiss. Anz. Ungar. Akad. Wiss. **61**, 478–495 (1942); Acta Univ. Szeged Sect. Sci. Math. **11**, 28–36 (1946); these Rev. **9**, 182; **8**, 263]. Using complex variable methods the author derives a rather complicated integral expression for the partial sums $S_n(x)$ of (*) which enables him to derive the following results. (i) $S_n(x)=S_{2n}^*(x)+\epsilon_n(x)$, where $S_{2n}^*(x)$ denotes the $(2n+1)$th partial sum of the ordinary Fourier series on $(-\pi, 3\pi)$ of the function $\Phi(x)=f(x)$ $(-\pi\leqq x<\pi)$, $\Phi(x)=-f(x-2\pi)$ $(\pi\leqq x<3\pi)$, and where $\epsilon_n(x)\to 0$ uniformly on $-\pi\leqq x\leqq\pi$. (ii) For any δ, $0<\delta<\frac{1}{2}$, there exists a series $\sum_{\nu=1}^{\infty}(A_\nu\cos\mu_\nu x+B_\nu\sin\mu_\nu x)$ satisfying $|A_\nu|+|B_\nu|=O(\nu^{-\delta})$, which is the formally differentiated series (*) of a continuous nondecreasing (Cantor type) function $F(x)$ on $(-\pi, \pi)$ and which converges to zero almost everywhere on $(-\pi, \pi)$. It is shown how this result disproves one of Fejes's results [second paper cited, p. 31] on the "almost uniqueness" of the non-harmonic series $\sum(a_\nu\cos\mu_\nu x+b_\nu\sin\mu_\nu x)$ having a given sum $s(x)$. *J. Korevaar* (Delft).

Baiada, Emilio. Un metodo di sommazione per le serie di funzioni ortonormali. Ricerche Mat. **1**, 107–123 (1952).

Let $\{\varphi_i\}$, $i=1, 2, \cdots$, be the Haar orthonormal set and let $\{\psi_i\}$ be an orthonormal set, defined everywhere on $[0, 1]$ and p.p. (almost everywhere) continuous and integrable. Let $f(x)$, integrable on $[0, 1]$, have p.p. a convergent development (a) $f(x)=\sum_{r=1}^{\infty}a_r\psi_r(x)$. The functions $f(x)$, $\psi_r(x)$ have p.p. convergent developments in the Haar functions: $f(x)=\sum_{i=1}^{\infty}b_i\varphi_i(x)$, $\psi_r(x)=\sum_{i=1}^{\infty}c_{ir}\varphi_i(x)$; and (a) can be rewritten as (b) $f(x)=\sum_{r=1}^{\infty}a_r\sum_{i=1}^{\infty}c_{ir}\varphi_i(x)$, convergent p.p. This suggests consideration of the double series (c) $\sum_{i,r=1}^{\infty}a_rc_{ir}\varphi_i(x)$. By (b), series (c) is p.p. summable by columns; and if $f\in L_2$ and $\{\psi_i\}$ is complete in L_2, then (c) converges p.p. by rows to $f(x)$. The above considerations suggest a method of summability for $\sum_{r=1}^{\infty}a_r\psi_r$: Given an orthonormal set $\{\psi_r\}$ and coefficients $\{a_r\}$, set up formally the double series (c). Let (L) be a method of summation of (c) according to some definite order in which the terms are taken (the simplest such would be summation by rows or by columns). Then the series $\sum_{r=1}^{\infty}a_r\psi_r(x)$ is said to be summable (L) if the series (c) converges by method (L). (The author also observes that one can introduce convergence factors $\lambda_{ir}(t)$, thus replacing (c) by $\lim_{t\to 0}\sum_{i,r=1}^{\infty}\varphi_i(x)\lambda_{ir}(t)c_{ir}a_r$.) Some remarks are made on this definition, and some results (which for brevity we omit) are obtained bearing on the convergence of series (c).

The following results are also shown: (i) If $f\in L_2$, its ψ_r-development $\sum_{r=1}^{\infty}a_r\psi_r(x)$ converges at x if and only if $f_m(x)\to 0$, where the Fourier coefficients $\{g_i\}$ of $f_m(x)$ relative to the Haar set $\{\varphi_r\}$ are defined by $g_i=\sum_{r=m}^{\infty}a_rc_{ir}$. (ii) Let us say that two orthonormal sets $\{\psi_r\}$, $\{\theta_r\}$ are equivalent in a class of functions (C) if whenever $f(x)\in C$ has a convergent development in either a ψ_r-series or a θ_r-series, the same is true of the other. Let $d_{rr}=\int_0^1\varphi_r\theta_r dx$. If the series $\sum_{r=1}^{\infty}r^2(1-d_{rr})$ converges, then $\{\varphi_r\}$, $\{\theta_r\}$ are equivalent in the class (L_2). *I. M. Sheffer* (State College, Pa.).

Vilenkin, N. Ya. On some almost orthogonal systems of functions. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 382–384 (1952). (Russian)

Denoting $2\sin x\sin[(m+1)x]$ by $P_m(x)$ and considering the two systems of functions of two variables, $0\leqq x\leqq\pi$, $0\leqq y\leqq\pi$: $P_m(x)P_n(y)$ and $Q_{mn}(x, y)$, where $Q_{00}=1$ and $Q_{mn}(x, y)=P_m''(x)P_n(y)+P_m(x)P_n''(y)$, the author proves that the first system is closed, while the second system becomes closed only if it is completed by adjoining to it the functions $p_k(x, y)$, $q_k(x, y)$, $p_k(y, x)$, $q_k(y, x)$, where $k=E(k)\geqq 1$, defined as follows:

$$4k\operatorname{ch}(\pi k/2)p_k(x, y)=\operatorname{sh}[k(\tfrac{1}{2}\pi-y)]\cos kx$$
$$4k\operatorname{sh}(\pi k/2)q_k(x, y)=\operatorname{ch}[k(\tfrac{1}{2}\pi-y)]\cos kx.$$

E. Kogbetliantz (New York, N. Y.).

Alexits, György. On the order of magnitude of approximations attainable by the series developments of theoretical physics and technology. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei **1**, 274–278 (1951). (Hungarian)

The author gives a brief introductory review of estimates for the remainder after n terms in the development in orthogonal functions of a function whose rth derivative satisfies a Lipschitz condition of order α. Estimates of the form const. $\times n^{-r-\alpha}\log n$ are noted in the Fourier case, and also for general orthogonal polynomials, the latter seemingly due to the author. Also surveyed is the converse problem.

F. V. Atkinson (Ibadan).

Boas, R. P., Jr. Sums representing Fourier transforms. Proc. Amer. Math. Soc. **3**, 444–447 (1952).

Let $\phi(x)$ be a given function and let $\Phi(x)$ be its Fourier transform, i.e., $\Phi(x)=\int_{-\infty}^{\infty}e^{-2\pi ixu}\phi(u)du$. Then Poisson's summation formula may be written in the form

$$(1)\quad x^{-1}e^{\pi i ab}\sum e^{2\pi i\alpha b}\phi[x^{-1}(n+b)]=e^{-\pi iab}\sum e^{2\pi i\alpha b}\Phi[x(n-a)].$$

Here $n=0,\pm1,\pm2,\cdots$ and a, b, and x are arbitrary real numbers. This note points out that (1) is formally equivalent to the statement: (2) $x^{-1}e^{\pi iab}\sum e^{2\pi i\alpha b}\phi[x^{-1}(n+b)]$ is the Fourier transform of $\sum e^{2\pi i\alpha b}|n-a|^{-1}\phi[x(n-a)^{-1}]$. Here $0<a<1$; if $a=0$, the first sum is to be diminished by $\int_{-\infty}^{\infty}\phi(u)du$ and the term with $n=0$ is omitted from the second sum. Suitable choices of a and b in (2) are shown to give rise to interesting special cases. The case $a=b=1/2$ has been studied by the reviewer [Bull. Amer. Math. Soc. **51**, 447–455 (1945); same Proc. **1**, 250–255 (1950); these Rev. **6**, 266; **11**, 592; **12**, 1002]. Conditions on ϕ were there given to insure the validity of (2). These conditions are actually less stringent than known conditions for the validity of Poisson's formula (1).

R. J. Duffin.

Ku, C. H., Yüh, M. I., and Chen, K. K. The abscissa of uniform convergence of a Laplace integral. J. London Math. Soc. **27**, 356–359 (1952).

The authors generalize Kuniyeda's formula for the abscissa of uniform convergence of a Dirichlet series [Tôhoku Math. J. **9**, 7–27 (1916)] to the Laplace-Stieltjes integral.

S. Agmon (Jerusalem).

Jaiswal, J. P. On Meijer transform. Math. Z. **55**, 385–398 (1952).

The Meijer transform is the integral transform

$$\Phi(s)=sF(s)=s\int_0^{\infty}e^{-st/2}W_{k+1/2,m}(st)(st)^{-k-1/2}f(t)dt,$$

where W is Whittaker's confluent hypergeometric function. The author gives the formal "rules" for this transform (which are the same for all integral transforms of the form $s\int_0^{\infty}K(st)f(t)dt$ and hence well known), writes down some transform pairs (paraphrases of known integrals), obtains several "theorems" which are immediate consequences of known properties (recurrence relations, multiplication theorems, and the like) of Whittaker's function, and gives examples.

A. Erdélyi (Pasadena, Calif.).

Jaiswal, J. P. A note on Meijer transform. Ann. Soc. Sci. Bruxelles. Sér. I. **66**, 55–60 (1952).

[See also the preceding review.] A further formula concerns a "chain" of two Laplace transformations followed by a Meijer transform. [Reviewer's remark: The function $\chi(p,s)$ occurring in the theorem may be expressed in terms of MacRobert's E-function.]

A. Erdélyi.

Olsen, Haakon. On a certain Hankel transform. Norske Vid. Selsk. Forh., Trondheim **24** (1951), 135 (1952).

Poli, L. Intégrales et calcul symbolique. Ann. Soc. Sci. Bruxelles. Sér. I. **66**, 21–26 (1952).

The author points out that in operational relations one can introduce arbitrary parameters. Multiplication by some function and integration leads to new operational formulas. He gives several examples.

A. Erdélyi.

Special Functions

Watson, G. N. Periodic sigma functions. Proc. London Math. Soc. (3) **2**, 129–149 (1952).

Let $\sigma(z)$ be the Weierstrass σ-function, admitting the periods $2\omega_1$ and $2\omega_3$ and let $\tau=\omega_3/\omega_1$. The author proves that there exists a countably infinite set of essentially distinct values such that $\sigma(z)$ is singly periodic if and only if τ belongs to that set; a similar property holds for the corresponding (Weierstrass) ζ-function. If $\sigma(z+\Omega)=\sigma(z)$ identically in z, it follows that $\Omega=2M\omega_1'+2N\omega_3'$, where $2\omega_1'$ and $2\omega_3'$ are some primitive periods. The corresponding periodicity factor of $\sigma(z)$ is

$$(-1)^{MN+M+N}\exp\{(2M\eta_1'+2N\eta_3')(z+M\omega_1'+N\omega_3')\}$$

and reduces identically to unity if and only if

$$2M\eta_1'+2N\eta_3'=0 \quad\text{and}\quad M\equiv N\equiv 0 \pmod 2.$$

In that case there exist primitive periods $2\omega_1$, $2\omega_3$, such that we have, for the associated Weierstrass constant, $\eta_1=(2M\eta_1'+2N\eta_3')/(M,N)=0$. This equation is solved, using the relations

$$2\eta_1\omega_1=\pi^2\left\{\tfrac{1}{3}-\sum_{r=1}^{\infty}4q^{2r}/(1-q^{2r})^2\right\}=-\vartheta_1'''(0,q)/6\vartheta_1'(0,q)$$

and series expansions for the theta-functions. For purely imaginary values of τ there exists only the solution $\tau/i=.52352$. Let now $D^{(0)}$ be the fundamental region of the modular group, defined by $|u|\leqq\frac{1}{2}$, $u^2+v^2\geqq 1$, $D^{(r)}$ those obtained from $D^{(0)}$ by integral translations, and let D stand for the region covered by all $D^{(r)}$; let $\Delta_{a/b}^{(r)}$ be the transform of $D^{(r)}$ by the modular substitution $T=(c+d\tau)/(a+b\tau)$, $ad-bc=1$, with cusp at $\tau=-a/b$, and let $\Delta_{-a/b}$ stand for the set of all triangles $\Delta_{a/b}^{(r)}$. Then, using properties of the modular group, Lagrange's theorem on the expansion of a root of Kepler's equation [Whittaker and Watson, Modern analysis, 4th ed., Cambridge Univ. Press, 1927, p. 132] and the method of majorant functions, it follows that, for every irreducible value m/n of $-a/b$, there exists one and only one root T in D, with the corresponding value of τ in $\Delta_{-a/b}$. If $|m|\leqq n/2$, then T is in $D^{(0)}$ and τ lies in $\Delta_{m/n}^{(0)}$. The paper also contains a complete tabulation of the values of T and τ, to 15 decimals, for all fractions m/n with $n\leqq 7$, $|m|\leqq n/2$, as well as many computational details worked out with the greatest care.

E. Grosswald (Philadelphia, Pa.).

Bochner, S. Theta relations with spherical harmonics. Proc. Nat. Acad. Sci. U. S. A. **37**, 804–808 (1951).

Hecke and Schoeneberg have extended the classical relation

$$\sum_{(n)}\exp\left(-\pi t\sum_{m=1}^{k}n_m^2\right)=t^{-k/2}\sum_{(n)}\exp\left(-\frac{\pi}{t}\sum_{m=1}^{k}n_m^2\right)$$

to

$$(1)\quad \sum_{(n)} P_g(n_1,\cdots,n_k)\exp\left(-\pi t\sum_{m=1}^{k} n_m^2\right) = i^g t^{-k/2-g}\sum_{(n)} P_g(n_1,\cdots,n_k)\exp\left(-\frac{\pi}{t}\sum_{m=1}^{k} n_m^2\right),$$

where (2) $P_g(x_1,\cdots,x_k)$ is any homogeneous polynomial of degree g which is a solution of the Laplacian $\Delta P=0$ in E_k [Hecke, Danske Vid. Selsk. Math.-Fys. Medd. **17**, no. 12 (1940); these Rev. **2**, 251; Schoeneberg, Math. Ann. **116**, 511–523, 700 (1939)]. In this paper the most comprehensive version of (1),

$$(3)\quad \sum_{(n)} \exp\left(-\pi t Q(n+\alpha)+2\pi i\sum_j (n_j+\alpha_j)\beta_j\right) = t^{-k/2}(\det Q)^{-1/2}\sum_{(n)}\exp\left(-\frac{\pi}{t}Q'(n+\beta)-2\pi i\sum_j n_j\alpha_j\right),$$

where $Q(t)=Q(t_1,\cdots,t_k)$ is an arbitrary real positive definite quadratic form and $Q'(t)$ is its "inverse", has been extended in a similar fashion to

$$(4)\quad \sum R_g(n+\alpha)\exp\left(-\pi t Q(n+\alpha)+2\pi i\sum_j (n_j+\alpha_j)\beta_j\right) = i^g t^{-k/2-g}(\det Q)^{-1/2}\sum R_g'(n+\beta)\times\exp\left(-\frac{\pi}{t}Q'(n+\beta)-2\pi i\sum_j n_j\alpha_j\right),$$

where $R_g(t)$ arises from a harmonic polynomial (2) by a substitution $x=At$ which transforms $\sum_j n_j^2$ into $Q(n)$, and $R_g'(t)$ arises by the inverse reciprocal substitution $x=(A^*)^{-1}t$. It suffices to prove

$$\int_{E_k} P_g(x_j)\exp\left(-\pi\sum_{m=1}^{k} x_m^2+2\pi i\sum_j x_j y_j\right)dv_x = i^g P_g(y_j)\exp\left(-\pi\sum_{m=1}^{k} y_m^2\right),$$

which follows by way of the relation

$$\int_0^\infty \exp\left(-p^2t^2\right)\{J_\nu(t)\}^{\nu+1}dt=(2p^2)^{-\nu-1}\exp\left(-1/4p^2\right)$$

from the theorem proved by the author:

$$\frac{1}{(2\pi)^{k/2}}\int_{E_k} P_g(x_j)\phi(|x|)\exp\left(i\sum x_j y_j\right)dv_x = \frac{i^g P_g(y_j)}{|y|^{k+2g}}\int_0^\infty \phi\left(\frac{t}{|y|}\right)J_{k+g}(t)t^{k+g+1}dt$$

whenever the left side is absolutely convergent. ($\phi(u)$ is supposed measurable in $0<u<\infty$, $|x|=(x_1^2+\cdots+x_k^2)^{1/2}$, $|y|=(y_1^2+\cdots+y_k^2)^{1/2}$, $l=\frac{1}{2}(k-2)$.) For $|y|\neq 0$ the significant relation

$$\lim_{\epsilon\to 0}\frac{1}{(2\pi)^{k/2}}\int_{E_k} P_g(x_j)|x|^{s-g}\exp\left(-\epsilon|x|+i\sum x_j y_j\right)dv_x = \frac{i^g P_g(y_j)}{|y|^{k+s+g}}\,\frac{2^{s+k/2}\Gamma\left(\frac{k+s+g}{2}\right)}{\Gamma\left(\frac{g-s}{2}\right)}$$

follows, where the constant s may be complex. For $s=-k/2$ the remarkable formula

$$\lim_{\epsilon\to 0}\frac{1}{(2\pi)^{k/2}}\int_{E_k}\frac{P_g(\xi)}{|x|^{k/2}}\exp\left(-\epsilon|x|+i\sum x_j y_j\right)dv_x=\frac{i^g P_g(\eta_j)}{|y|^{k/2}},$$

which assigns to the Fourier transform peculiar eigenfunctions with eigenvalues i^g. The paper terminates with a remark on the role of polynomials which are harmonic versus those which are not. *S. C. van Veen* (Delft).

Burchnall, J. L. Some determinants with hypergeometric elements. Quart. J. Math., Oxford Ser. (2) **3**, 151–157 (1952).

For any homogeneous isobaric invariant $I(a_0,\cdots,a_n)$ of the binary n-ic $\sum\binom{n}{r}a_r X^{n-r}Y^r$, if the coefficient a_r is replaced by $U_r^\lambda(x)=\binom{r+2\lambda-1}{r}^{-1}P_r^\lambda(x)$, where $P_r^\lambda(x)$ is the ultraspherical polynomial of degree r and parameter λ, the author has shown [Proc. London Math. Soc. (3) **1**, 232–240 (1951); these Rev. **13**, 648] that $I\{U_r^\lambda(x)\}=C(\lambda)(x^2-1)^{\frac{1}{2}w}$, where $C(\lambda)$ is a constant depending on n, and w is the weight of the invariant. An application of this principle to Hankel determinants gives

$$\det\left[U_{r+s}^\lambda(x)\right]=C(\lambda)(x^2-1)^{\frac{1}{2}n(n+1)},$$

which was independently established by the reviewer, Seidel and Szász [Duke Math. J. **18**, 1–10 (1951); these Rev. **12**, 702], where $C(\lambda)$ was expressed in the form of a multiple definite integral. The author now by two different methods obtains an explicit formula for $C(\lambda)$. *E. F. Beckenbach.*

Thiruvenkatachar, V. R., and Nanjundiah, T. S. Inequalities concerning Bessel functions and orthogonal polynomials. Proc. Indian Acad. Sci., Sect. A. **33**, 373–384 (1951).

Improving a result of Szász [Proc. Amer. Math. Soc. **1**, 256–267 (1950); these Rev. **11**, 662], the authors prove that, for $-1\leqq x\leqq 1$, $(1-x^2)/2n\leqq P_n^2(x)-P_{n-1}(x)P_{n+1}(x)<2/\pi n$; here $P_n(x)$ is, as usual, Legendre's polynomial and the constant $2/\pi$ can not be replaced by a smaller one. These inequalities are connected with one of Turán which had an extraordinary resonance in recent literature. Analogous inequalities are discussed for the Bessel functions J_ν and I_ν. Finally new proofs are given for certain known inequalities of the Turán type [Szegö, Bull. Amer. Math. Soc. **54**, 401–405 (1948); these Rev. **9**, 429] involving ultraspherical and Laguerre polynomials. *G. Szegö.*

Landauer, R. Associated Legendre polynomial approximations. J. Appl. Phys. **22**, 87–89 (1951).

An approximate expression for the associated Legendre polynomial $P_l^m(\cos\theta)$ is obtained by a phase integral method, which is an extension of the WBK method. Here m and l are integers; $l>m\geqq 0$; $\sin\theta_0=m/(l+\frac{1}{2})$. For

$$\theta_0<\theta<\pi-\theta_0,\quad P_l^m(\cos\theta)\sim\frac{C}{(\sin^2\theta-\sin^2\theta_0)^{1/4}}\cos\left(\vartheta(\theta)-\frac{\pi}{4}\right),$$

$$\theta<\theta_0,\quad P_l^m(\cos\theta)\sim\frac{1}{2}\,\frac{C}{(\sin^2\theta_0-\sin^2\theta)^{1/4}}e^{-d(\theta)},$$

$$\theta>\pi-\theta_0,\quad P_l^m(\cos\theta)\sim(-1)^{l-m}\frac{C}{(\sin^2\theta_0-\sin^2\theta)^{1/4}},$$

where

$$\vartheta(\theta)(l+\tfrac{1}{2})\cos^{-1}\left(\frac{\cos\theta}{\cos\theta_0}\right)-m\cos^{-1}\left(\frac{\tan\theta_0}{\tan\theta}\right),$$

$$d(\theta)m\cosh^{-1}\left(\frac{\tan\theta_0}{\tan\theta}\right)-(l+\tfrac{1}{2})\cosh^{-1}\left(\frac{\cos\theta}{\cos\theta_0}\right),$$

$$C=\left[\frac{4}{\pi(2l+1)}\,\frac{(l+m)!}{(l-m)!}\right]^{\frac{1}{2}}.$$

S. C. van Veen (Delft).

Kuhn, T. S. A convenient general solution of the confluent hypergeometric equation, analytic and numerical development. Quart. Appl. Math. **9**, 1–16 (1951).

Whittaker's standard form of the confluent hypergeometric equation

$$\frac{d^2U}{dy^2}+\left[-\frac{1}{4}+\frac{n}{y}+\frac{1-4m^2}{4y^2}\right]U=0$$

is transformed by the substitution $r=ny/2$, $m=l+\frac{1}{2}$ to the form

$$(1)\qquad \frac{d^2U^{(l,n)}}{dr^2}+\left[-\frac{1}{n^2}+\frac{2}{r}-\frac{l(l+1)}{r^2}\right]U^{(l,n)}=0.$$

The further substitutions $z=(8r)^{\frac{1}{2}}$, $U^{(l,n)}=\frac{1}{2}zV^{(l,n)}$ reduce (1) to the form

$$(2)\qquad \left[z^2\frac{d^2}{dz^2}+z\frac{d}{dz}+z^2-(2l+1)^2\right]V^{(l,n)}-n^{-2}(\tfrac{1}{2}z)^4V^{(l,n)}=0.$$

It is at first assumed, and proved in the appendix, that the general solution of (2) may be written in the form of a power-series in $1/n^2$, i.e.,

$$(3)\qquad V^{(l,n)}(z)=\sum_{k=0}^{\infty}n^{-2k}V_k^{(l)}(z),$$

where the functions $V_k^{(l)}(z)$ are analytic functions of z and the series (3) converges absolutely and uniformly for all real l and for all real n in $|n|\geqq n_0$ (n_0 arbitrary >0). By equating the various powers of $1/n^2$ an infinite set of simultaneous differential equations for $V_k^{(l)}(z)$ is obtained. The first, $\{z^2d^2/dz^2+zd/dz+z^2-(2l+1)^2\}V_0^{(l)}=0$, is Bessel's equation of order $2l+1$. The general solution is the cylindrical function $C_{2l+1}(z)=aJ_{2l+1}(z)+bY_{2l+1}(z)$ (a and b arbitrary).

Tables I and II list the formulas for $U_k^{(l)}(z)$ with $k=0$ through 7, $l=0$ and 1. Tables III and IV list corresponding formulas for the functions $D_k^{(0)}(z)$ and $D_k^{(l)}(z)$ where $D_k^{(l)}(z)=\frac{1}{2}zdU_k^{(l)}(z)/dz$. Tables V–VIII list values of the functions ${}^0U_k^{(l)}(z)$, ${}^1U_k^{(l)}(z)$, ${}^0D_k^{(l)}(z)$, and ${}^1D_k^{(l)}(z)$ for $l=0$ and 1, $k=0$ through 7, and $z=3.5(0.5)7.5$; where a superscript zero preceding the function indicates substitution of J_n for C_n, and a superscript one connotes use of Y_n. It can be shown that ${}^0U^{(l,n)}(z)$ is the only particular solution of (1) with a zero at the origin. Therefore

$${}^0U^{(l,n)}(z)=[n^{l+1}/\Gamma(2l+2)]M_{n,l+\frac{1}{2}}(z^2/4n).$$

A second solution of physical interest, the only particular solution which $\to 0$ for $z\to\infty$, is noted. An alternate generating procedure is given, where C_{2l+1} is expressed by C_0 and C_1. *S. C. van Veen* (Delft).

Olver, F. W. J. Some new asymptotic expansions for Bessel functions of large orders. Proc. Cambridge Philos. Soc. **48**, 414–427 (1952).

Asymptotic expansions for the first and second Hankel function $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$ for large values of z, which are valid in the transitional region, i.e., when the variable z is of the form $z=\nu+\tau\nu^{1/3}$ (τ any fixed number) are derived. Starting with the integral representation of Sommerfeld's type for the Hankel functions and using a variation of the method of steepest descents, the following asymptotic expansions are established

$$(1)\quad H_\nu^{(1)}(\nu+\tau\nu^{1/3})\sim 2^{1/3}\nu^{-1/3}[\mathrm{Ai}\,(-2^{1/3}\tau)-i\,\mathrm{Bi}\,(-2^{1/3}\tau)]\sum_{s=0}^{\infty}A_s(\tau)\nu^{-2s/3}$$
$$+2^{2/3}\nu^{-1/3}[\mathrm{Ai}'\,(-2^{1/3}\tau)-i\,\mathrm{Bi}'\,(-2^{1/3}\tau)]\sum_{s=0}^{\infty}B_s(\tau)\nu^{-2s/3},$$

$$(2)\quad H_\nu^{(2)}(\nu+\tau\nu^{1/3})\sim 2^{1/3}\nu^{-1/3}[\mathrm{Ai}\,(-2^{1/3}\tau)+i\,\mathrm{Bi}\,(-2^{1/3}\tau)]\sum_{s=0}^{\infty}A_s(\tau)\nu^{-2s/3}$$
$$+2^{2/3}\nu^{-1/3}[\mathrm{Ai}'\,(-2^{1/3}\tau)+i\,\mathrm{Bi}'\,(-2^{1/3}\tau)]\sum_{s=0}^{\infty}B_s(\tau)\nu^{-2s/3}.$$

Ai (x) and Bi (x) are the Airy functions and $A_s(\tau)$ and $B_s(\tau)$ are polynomials in τ. The values of $A_0(\tau)$, $\cdots$, $A_4(\tau)$, $B_0(\tau)$, $\cdots$, $B_4(\tau)$ are given. (1) and (2) are derived under the assumption $|\arg\nu|<\pi/2$, but it is shown later that (1) is valid throughout the range $-\pi/2<\arg\nu<3\pi/2$ and (2) throughout the range $-3\pi/2<\arg\nu<\pi/2$. Similar expansions are given for $J_\nu(\nu+\tau\nu^{1/3})$ and $Y_\nu(\nu+\tau\nu^{1/3})$.

F. Oberhettinger (Washington, D. C.).

Harmonic Functions, Potential Theory

Royden, H. L. On the regularity of boundary points in potential theory. Proc. Amer. Math. Soc. **3**, 82–86 (1952).

L'auteur démontre le théorème suivant: un point-frontière P_0 d'un domaine G à 3 dimensions est "régulier" vis-à-vis du problème de Dirichlet s'il existe une sphère S de centre P_0 telle que les points de S qui peuvent être joints à P_0 par un segment extérieur à G forment un ensemble de capacité logarithmique positive. Ce critère comprend en particulier celui de Evans dans l'oeuvre analysé ci-dessous: P_0 est régulier s'il existe un triangle ayant un sommet en P_0 qui n'appartient pas à G. La démonstration est basée sur la construction d'une "barrière" pour P_0, construction basée sur deux lemmes plans. *R. de Possel* (Alger).

Evans, Griffith C. Lectures on multiple valued harmonic functions in space. Univ. California Publ. Math. (N.S.) **1**, 281–340 (1951).

L'auteur étudie (après quelques précurseurs qu'il cite comme Sommerfeld) les fonctions harmoniques dans une variété euclidienne connexe à trois dimensions, analogue aux surfaces de Riemann élémentaires à un nombre fini de feuillets. Les espaces-feuillets composants sont reliés par des chemins autour de courbes simples fermées (de capacité nulle), qui servent à limiter des surfaces à plusieurs feuillets jouant le rôle des coupures des surfaces de Riemann. Après quelque étude topologique, l'auteur résout pour un domaine borné le problème de Dirichlet généralisé (à donnée-frontière continue), introduit la fonction de Green (pour un domaine quelconque) et la fonction harmonique qui tend vers 1 à l'infini dans un feuillet et vers 0 dans les autres; il étend les formules de Green. Ces espaces à feuillets sont utiles pour résoudre des problèmes de l'espace ordinaire comme la recherche, développée ailleurs par l'auteur, de la surface de capacité minima limitée par une courbe donnée. On s'occupe ici d'un problème de ce type en caractérisant dans l'espace E à deux feuillets la surface $K(AM)=1/2\overline{AM}$ (K fonction de Green dans E, de pôle A; $\overline{AM}$ distance des projections dans l'espace euclidien support). Signalons que le Ref. et Choquet viennent depuis d'étudier les fonctions harmoniques dans des espaces plus généraux [Ann. Inst. Fourier Grenoble **3**, 199–263 (1952)]. *M. Brelot.*

Neustadter, Siegfried F. Multiple valued harmonic functions with circle as branch curve. Univ. California Publ. Math. (N.S.) **1**, 397–432 (1951).

Illustrant la théorie de Evans de l'article analysé ci-dessus, l'auteur explicite et étudie la fonction de Green des espaces localement euclidiens à trois dimensions avec un nombre fini ou non d'espaces-feuillets et un cercle comme courbe de ramification. Il exprime $1/r$ au moyen des coordonnées toroïdales et forme des intégrales qui fournissent la solution d'après des méthodes de Sommerfeld et Hobson. *M. Brelot* (Grenoble).

Jenkins, James A., and Morse, Marston. Contour equivalent pseudoharmonic functions and pseudoconjugates. Amer. J. Math. **74**, 23–51 (1952).

The authors consider pseudoharmonic functions (functions locally topologically equivalent to harmonic functions) in a Jordan domain D of the plane; boundary conditions are imposed, which effectively permit continuation to a larger domain. Two such functions u, v are termed contour equivalent if their level-curve families are topologically equivalent under an orientation-preserving homeomorphism of $\bar{D}$; if the homeomorphism actually transforms u into v, the contour equivalence is termed strict. By the net of u is meant the set formed by the level curves through multiple points. It is proved that two functions whose nets are homeomorphic under an orientation-preserving homeomorphism of $\bar{D}$ are contour equivalent; a stronger form of this result is given, involving only inner properties of the net. It is shown that each u is contour equivalent to a v whose net is formed of hyperbolic lines in the unit circle. It is proved that each function u has a pseudoconjugate v, so that $u+iv$ defines a light interior mapping. It is proved that every u is strictly contour equivalent to a harmonic function v. These results are very closely related to those of Boothby [same J. **73**, 405–438, 512–538 (1951); these Rev. **13**, 149, 266] to which no reference is given; in particular, the existence of the pseudoconjugate is established by Boothby without any assumptions about behavior on the boundary; also Boothby shows, in this more general case, that each u is strictly contour equivalent to a harmonic function. *W. Kaplan.*

Green, John W. On the level surfaces of potentials of masses with fixed center of gravity. Pacific J. Math. **2**, 147–152 (1952).

Précisant un résultat proposé en exercice par Kellogg [Foundations of potential theory, Springer, Berlin, 1929, p. 144], l'auteur détermine le "défaut de sphéricité" maximum des surfaces de niveau, extérieures à la sphère-unité K, du potentiel newtonien engendré par une distribution positive portée par K et ayant l'origine pour centre de gravité. La méthode utilisée s'applique aux potentiels en $\varphi(r)$, avec φ convexe et décroissante. *J. Deny* (Strasbourg).

Minasyan, R. S. On a mixed boundary problem for the Laplace equation for a rectangle. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 293–304 (1952). (Russian)

Solution of the problem in question, supposed to be axially symmetric, by means of suitable series. Numerical results. *L. Gårding* (Lund).

Aržanyh, I. S. Resolvents of basic problems in field theory. Doklady Akad. Nauk SSSR (N.S.) **82**, 545–548 (1952). (Russian)

In a region Q the vector $\mathbf{v}$ is to have prescribed curl and divergence, and on S, the boundary of Q, is to have either given tangential or given normal component. In addition, the data are subject to solubility conditions and for the exterior problem there appears to be a condition at infinity. The author puts $\mathbf{v}=\mathbf{v}_0+\text{grad}\,\Phi$, where $\mathbf{v}_0$ has the required curl and divergence and Φ is to be harmonic. It is shown how the choice of $\mathbf{v}_0$, and then Φ from the boundary conditions, may be made to depend on known integral equations and boundary problems of potential theory. *F. V. Atkinson.*

Aržanyh, I. S. Integral representation of a vector field. Doklady Akad. Nauk SSSR (N.S.) **85**, 55–58 (1952). (Russian)

The author is concerned with the same boundary problems as in the paper reviewed above. From a vector integral identity he deduces integral equations for the unknown normal or tangential components of $\mathbf{v}$ on S. These equations are simplified for special problems of the motion of a body through fluid, of elasticity, and of electrodynamics. There are no details of the derivation of the integral equations, which would appear to be singular, the kernels involving higher derivatives of Green's functions. *F. V. Atkinson.*

Miranda, Carlo. Sull'integrazione delle forme differenziali esterne in n variabili di grado $n-1$ e sul lemma di Haar per gli integrali multipli. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **85**, 246–254 (1951).

This paper is concerned with the integration of certain systems of partial differential equations in which the integrability conditions are given in integral (rather than differential) form. For simplicity in this review, we restrict ourselves to the "three-dimensional" case, although the author's discussion is always in terms of m-dimensions. The fundamental theorem asserts the existence throughout a sufficiently regular region T of a vector solution $u(x, y, z)$ of the system curl $u=f$ where $f=f(x, y, z)$ is a known vector function satisfying a Hölder condition (but not, of course, necessarily differentiable) and satisfying also the condition that the surface integral $\int\int(f\cdot n)ds=0$ when extended over arbitrary spheres in T. Here n denotes the unit exterior normal vector to the element of area ds. The author's general proof is based entirely on ingenious applications of well-known theorems in the Newtonian potential theory of m-dimensions. *D. C. Lewis* (Baltimore, Md.).

Differential Equations

Germay, R. H. Extension d'un théorème de Poincaré aux équations récurro-différentielles de forme normale dépendant d'un paramètre variable. Bull. Soc. Roy. Sci. Liège **20**, 678–684 (1951).

By a classical method, involving the use of dominating functions, the author proves an existence and uniqueness theorem for solutions of a system of differential equations of the form

$$\frac{dx_n}{dt}=f_n(t, x_n, x_{n+1}, \lambda) \quad (n=1, 2, 3, \cdots).$$

The functions $f_n(t, u, v, \lambda)$ are assumed to be representable by power series in u, v, λ, with coefficients which are continuous functions of t, in a region defined by relations of the form

$$|u|\leqq\rho, \quad |v|\leqq\rho, \quad |\lambda|\leqq\rho, \quad t_0\leqq t\leqq t_1.$$

It is also assumed that the functions $|f_n|$ possess a common upper bound in this region. *L. A. MacColl.*

Germay, R. H. Extension d'un théorème de Poincaré aux systèmes d'équations récurro-différentielles de forme normale dépendant d'un paramètre variable. Ann. Soc. Sci. Bruxelles. Sér. I. 66, 27–30 (1952).

The results of the paper described in the preceding review are extended to the case of a system of differential equations of the form

$$\frac{dx_n}{dt}=F_n(t, x_n, y_n, z_n, x_{n+1}, y_{n+1}, z_{n+1}, \lambda),$$

$$\frac{dy_n}{dt}=G_n(t, x_n, y_n, z_n, x_{n+1}, y_{n+1}, z_{n+1}, \lambda),$$

$$\frac{dz_n}{dt}=H_n(t, x_n, y_n, z_n, x_{n+1}, y_{n+1}, z_{n+1}, \lambda),$$

$$(n=1, 2, 3, \cdots).$$

L. A. MacColl (New York, N. Y.).

Hayashi, Kyuzo, and Yoshizawa, Taro. New treatise of solutions of a system of ordinary differential equations and its application to the uniqueness theorems. Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. 26, 225–233 (1951).

The authors consider the system of ordinary differential equations

$$(1)\qquad \frac{dy_i}{dx}=f_i(x, y_1, \cdots, y_n) \quad (i=1, 2, \cdots, n)$$

in the domain G defined by $0\leqq x\leqq a$, $|y_i|\leqq b_i$. The assumptions on (1) are those of Carathéodory [Vorlesungen über reelle Funktionen, 2. Aufl., Teubner, Leipzig-Berlin, 1927, pp. 665–672]. With the notation $y=(y_1\cdots y_n)$ let $P=(x_P, y_P)$, $Q=(x_Q, y_Q)$ be points of G $(x_Q>x_P)$. Let $\mathfrak{Y}_{PQ}$ be the set of all functions $y(x)$ which are absolutely continuous in $x_P\leqq x\leqq x_Q$ and such that the curve $y=y(x)$ lies in G and contains P and Q. Define

$$D(P, Q)=\inf_{y(x)\in\mathfrak{Y}_{PQ}}\int |y'(x)-f(x, y)|dx \quad \text{if} \quad x_Q>x_P$$
$$=|y_P-y_Q| \quad \text{if} \quad x_Q=x_P$$
$$=D(Q, P) \quad \text{if} \quad x_Q<x_P.$$

The main theorem of the paper is then that $D(P, Q)=0$ is the necessary and sufficient condition for Q and P to be on the same solution of (1). In addition some other properties of $D(P, Q)$ are proved and, as an application, a uniqueness theorem for (1) is stated. *E. H. Rothe.*

Viswanatham, B. On the existence of a solution of an infinite differential system. J. Indian Math. Soc. (N.S.) 16, 13–24 (1952).

An existence theorem is proved for an infinite system of differential equations of the form $dy_i/dx=f_i(x, y_1, y_2, \cdots)$, where $i=1, 2, 3, \cdots$, under the assumption that numbers $M_1, M_2, \cdots$ exist such that $|f_i|\leqq M_i$ for $a\leqq x\leqq b$ and for all values of the y's. It is also assumed that the f's are continuous in the sense that for every sequence in which $\lim_{n\to\infty} y_{pn}=y_p$ $(p=1, 2, 3, \cdots)$ we always have $\lim_{n\to\infty} f_i(x, y_{1n}, y_{2n}, y_{3n}, \cdots)=f_i(x, y_1, y_2, y_3, \cdots)$. The proof involves the use of Ascoli's theorem and an additional application of Cantor's principle of diagonal selection in a more or less familiar manner. Unfortunately the definition of continuity is a highly restrictive one, under which, for example, the function

$$f(y_1, y_2, \cdots)=y_1^2+y_2^2+\cdots,$$

when $y_1^2+y_2^2+\cdots$ converges to a value less than 1 and $f(y_1, y_2, \cdots)=1$ otherwise, would not be continuous at $0, 0, \cdots$. *D. C. Lewis* (Baltimore, Md.).

Los', F. S. On the principle of averaging for differential equations in Hilbert space. Ukrain. Mat. Žurnal 2, no. 3, 87–93 (1950). (Russian)

This is the generalization to equations in Hilbert space H of the time averaging process of Kryloff and Bogoliuboff [Introduction to non-linear mechanics, Princeton Univ. Press, 1943; these Rev. 4, 142]. The basic equation is

$$(1)\qquad \dot{x}=\epsilon X(t, x)$$

where x is a Hilbert space vector whose components $x_n(t)$ satisfy

$$\sum x_n^2<\infty$$

and X is uniformly bounded in some domain $D\subset H$. Moreover for x', $x''\in D$ it is supposed that

$$|X(t, x')-X(t, x'')|<\lambda|x'-x''|$$

for all $t\geqq 0$. Finally one assumes that there exists an $X_0(x)$ such that uniformly with respect to x in D and as $T\to+\infty$:

$$\frac{1}{T}\int_0^T X(\tau, x)d\tau-X_0(x)\to 0.$$

Theorem. Under the above assumptions if $\xi(t)$ is a solution of $\dot{\xi}=\epsilon X_0(\xi)$ defined for all $t\geqq 0$ and in D together with its ρ neighborhood, and if $x(t)$ is the solution of (1) such that $x(0)=\xi(0)$ (it is obtainable in the usual manner), then for arbitrary η, ρ, $1/L>0$, there exists an $\epsilon_0>0$ such that for $0<\epsilon<\epsilon_0$ and $0<t<L/\epsilon$, there takes place: $|x(t)-\xi(t)|<\eta$. *S. Lefschetz* (Princeton, N. J.).

Persidskiĭ, K. On the stability of solutions of denumerable systems of differential equations. Izvestiya Akad. Nauk Kazah. SSR. 56, Ser. Mat. Meh. 2, 3–35 (1948). (Russian. Kazak summary)

Let there be given a system

$$(1)\qquad \dot{x}=\omega(t, x)$$

where x and ω are vectors with a countable number of components, $t\geqq 0$, and the components x_s of x are bounded: $|x_s|\leqq R$. Moreover (1) satisfies a Lipschitz-like condition

$$|\omega_s(t, x')-\omega_s(t, x'')|$$
$$\leqq A(t)\beta_s|x_1'-x_1''|+\cdots+\beta_1|x_s'-x_s''|$$
$$+\alpha_1|x_{s+1}'-x_{s+1}''|+\cdots,$$

where $A(t)$ is positive and continuous for $t\geqq 0$ and the α_s, β_s are positive and such that $\beta_1+\beta_2+\alpha_2+\cdots+\beta_m+\alpha_m+\cdots$ is convergent. Furthermore, (a) $\omega(t, 0)=0$; (b) $|\omega_s(t, x)|<B(t)$, for all bounded x_s, where $B(t)$ is continuous for $t\geqq 0$; (c) for every sequence $u_1(t)$, $u_2(t)$, $\cdots$, of functions continuous on a certain segment $[a, b]$ the functions $\omega_s(t, u(t))$ are measurable. Under the circumstances the author develops for the system (1) the general theory of Lyapunov (first and second methods). *S. Lefschetz* (Princeton, N. J.).

Persidskiĭ, K. On an estimate for characteristic values. Izvestiya Akad. Nauk Kazah. SSR. 56, Ser. Mat. Meh. 2, 36–45 (1948). (Russian. Kazak summary)

Let the system

$$\dot{x}_s=\sum p_{sr}x_r, \quad s, r=1, 2, \cdots, n,$$

be such that the coefficients p_{sr} are continuous and periodic with period ω. Let $\{x_{sr}\}$ be the system of solutions assuming for $t=0$ the initial values given by the unit matrix of order

n. Set

(1) $$\gamma_m^2=\sum_{r,s}|x_{sr}(\gamma_m\omega)|$$

where n is a positive integer. Theorem. $(-1/m\omega)\log\gamma_m$ is a lower bound for the characteristic numbers of (1). Further consequences and analogous results are derived in the paper.

S. Lefschetz (Princeton, N. J.).

Goršin, S. On stability of motion with constantly acting disturbances. Izvestiya Akad. Nauk Kazah. SSR. 56, Ser. Mat. Meh. 2, 46–73 (1948). (Russian. Kazak summary)

Goršin, S. Critical cases. Izvestiya Akad. Nauk Kazah. SSR. 56, Ser. Mat. Meh. 2, 74–101 (1948). (Russian. Kazak summary)

Goršin, S. On the stability of the solutions of a denumerable system of differential equations with constantly acting disturbances. Izvestiya Akad. Nauk Kazah. SSR. 60, Ser. Mat. Meh. 3, 32–38 (1949). (Russian. Kazak summary)

Let there be given a system

(1) $$\dot{z}_k=Q_k(z,t)+R_k(z,t),\quad k=1,2,\cdots,n,$$

where for $t\geqq 0$, Q_k, R_k are continuous, positive in a certain real or complex domain H and have a unique Cauchy integral there. Let $z_k=\varphi_k(t)$ be a solution of (1) and $z_k=\psi_k(t)$ a solution of the analogous system with the R_k replaced by zeros. It is assumed that $\psi\in H$ for all $t\geqq 0$. Consider now n functions $F_r(z,t)$ continuous and single-valued together with all their partials as to the z_k, in H and this for all $t\geqq 0$. It is assumed furthermore that the Jacobian

$$\frac{D(F_1,\cdots,F_n)}{D(z_1,\cdots,z_n)}\neq 0$$

and that the partials $\partial F_r/\partial x_s$ are uniformly bounded in H. Let finally $x_s=F(z,t)-F(\psi,t)$, $z=\varphi(t)$. Given ϵ does there exist σ, ρ such that if $t_0\geqq 0$ and

$$|\psi_k(t_0)-\varphi_k(t_0)|<\sigma,\quad |R_k(z,t_0)|<\rho,$$

then $|x_k|<\epsilon$ for $t\geqq t_0$. If so, the system (1) is said to be stable relative to the functions F for positive perturbations, otherwise it is said to be unstable in the same sense. In the first two papers the author extends more or less to this type of stability the general results of Lyapunov. In the third paper he discusses the extension to countable systems patterned after those of Persidskiĭ [see the second preceding review].

S. Lefschetz (Princeton, N. J.).

Rešetov, M. R. On the stability of the solutions of a denumerable system of differential equations, the linear parts of which have triangular form. Izvestiya Akad. Nauk Kazah. SSR. 60, Ser. Mat. Meh. 3, 39–76 (1949). (Russian. Kazak summary)

Harasahal, V. On stability in the first approximation of the solutions of denumerable systems of differential equations. Izvestiya Akad. Nauk Kazah. SSR. 60, Ser. Mat. Meh. 3, 77–84 (1949). (Russian. Kazak summary)

These two papers may be considered as two parts of the same paper. For the purpose of this review the numbering will be consecutive.

First paper. The author considers first a system

(1) $$\dot{x}_s=\sum_{k=1}^{\infty}p_{sk}x_k;\quad s=1,2,\cdots,$$

with coefficients p_{sk} real or complex, continuous in t for $t>0$ and such that

(2) $$\sum_{k=1}^{s-1}|p_{sk}|<a,\quad |p_{ss}|<L(t),\quad s=1,2,\cdots,$$

where a is a fixed positive constant and $L(t)$ continuous in t for $t>0$. Theorem 1. (a) If for all $t>0$

(3) $$\left|\exp\int_0^t p_{ss}(\tau)d\tau\right|\leqq N,$$
$$\left|\exp\int_0^t p_{ss}(\tau)d\tau\right|\cdot\int_0^t\left|\exp-\int_0^\tau p_{ss}(t')dt'\right|d\tau\leqq M,$$
$$s=1,2,\cdots,$$

(4) $$aM\leqq\theta<1,$$

where M, N are certain fixed constants, then the solution $x=0$ of (1) is asymptotically stable. (b) Same conclusion if the first inequality (3) holds and for some positive t

(5) $$-p_{ss}(t)>\gamma>0;\qquad (b)\quad a/\gamma=\alpha M_1<\theta<1.$$

Consider now the system

(6) $$\dot{x}_s=\sum p_{sk}x_k+L_s(t,x),\quad L_s(t,0)=0,$$

and let H denote the region $t\geqq 0$, $|x_s|<b$, $s=1,2,\cdots$. Theorem 2. Let the p_{sk}, L_s, be continuous in H and let (2), (3), (4) hold. Let further in H

(7) $$|L_s(t,x)|<\frac{(1-\alpha)(1-\theta)b}{M},\quad 0<\alpha<1,\ t\geqq 0,$$

where α is a constant and for any two points (t,x'), $(t,x'')\in H$ let

(8) $$\left|L_s(t,x')-L_s(t,x'')\right|\leqq\frac{(1-\alpha)(1-\theta)}{MB}\sum\beta_k|x_k'-x_k''|,$$

where the β_k are positive constants such that $B=\sum\beta_k$ converges. If the initial values $x_s(0)$ are such that

$$|x_s(0)|<\frac{b\alpha}{N}(1-\theta),$$

then the solution $(t,x(t))$ of (6) remains in H. The author considers also the stability of (1) when $aM=1$. Returning to (1) the author proves: Theorem 3. Still assume (2) if $\exp\int_0^t p_{ss}(\tau)d\tau$ is unbounded; then the solution $x=0$ of (1) is unstable.

Second paper. The author investigates for stability a system (6) in which the L_s are of higher order than unity near the origin. That is to say in a suitable region H

(9) $$|L_s(t,x)|<DX^\beta,$$

where $D>0$, $\beta>1$, and $X=\sup\{|x_s|\}$. It is moreover assumed that: (a) $|p_{s1}|+\cdots+|p_{ss}|<A$, $s=1,2,\cdots$, where A is a fixed constant; (b) given any $\delta>0$, there exists a $T=T(\delta)>0$ such that, whatever $t>T$, the inequality $\sum|p_{sk}-a_{sk}|<\delta$, holds where the a_{sk} are suitable constants. Under these conditions the author derives stability theorems analogous to those of Lyapunov.

S. Lefschetz.

Malkin, I. G. On a problem of the theory of stability of systems of automatic regulation. Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 365–368 (1952). (Russian)

The following problem was proposed by Alzerman [Uspehi Matem. Nauk 4, no. 4(32), 187–188 (1949); these Rev. 11, 177]. Let the system of order n

(1) $$\dot{x}_1=\sum a_{1i}x_i+f(x_k)$$
$$\dot{x}_s=\sum a_{si}x_i,\quad s=2,3,\cdots,n,$$

where the a_{ji} are constants represent an automatic regulator. Let the system such as (1) but with $f(x_k)$ replaced by hx_k have all characteristic roots with negative real parts for all h such that $0<\alpha<h<\beta$. Assuming that $f(0)=0$ and that for all $x_k\neq 0$: $\alpha x_k^2<f(x)<\beta x_k^2$, is it true that $x=0$ will be asymptotically stable for (1) whatever the initial disturbance? In the present paper this is shown to hold for the two systems

$$\dot{x}=ay+f(x),\quad \dot{y}=bx+cy, \tag{2}$$

$$\dot{x}=ax+f(y),\quad \dot{y}=bx+cy, \tag{3}$$

under the following assumptions: System (2). Setting $f(x)=xh(x)$ then for $|x|$ sufficiently large: $ch(x)>ab$. System (3). Setting $f(y)=yh(y)$ then for $|y|$ sufficiently large: $ac>bh(y)$. The proofs are based on the use of the function V of Lyapunov. [Relevant references: Erugin, Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 459–512, 659–664 (1950); these Rev. **12**, 412]. *S. Lefschetz.*

Barbuti, Ugo. Sulla stabilità delle soluzioni per la equazione: $x''+B(t)x=0$. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 170–175 (1952).

The author uses the result that $x_1(t)=e^u\sin v$, $x_2(t)=e^u\cos v$ are solutions of $x''+a(t)x=0$ if $v=b^2e^{-2u(s)}ds+c$, and $a(t)=-u''-u'^2+b^2e^{-4u}$, to derive some results concerning the boundedness of the solutions of (1) $x''+(b^2+c(t))x=0$. In particular he demonstrates that if $c=f(t)g(t)$ where f is monotone and square integrable and $\int_s^t g(t)\sin 2bt\,dt$, $\int_s^t g(t)\cos 2bt\,dt$ are bounded for all s and $t\geqq t_0$, then all solutions of (1) are bounded. This generalizes a result due to Ascoli [same Rend. (8) **9**, 129–134 (1950); these Rev. **12**, 705] and Prodi [ibid. **10**, 447–451; **11**, 30–34 (1951); these Rev. **13**, 347, 653]. *R. Bellman* (Santa Monica, Calif.).

Miller, J. C. P. On the choice of standard solutions to Weber's equation. Proc. Cambridge Philos. Soc. **48**, 428–435 (1952).

In Quart. J. Mech. Appl. Math. **3**, 225–235 (1950) [these Rev. **12**, 29] the author discussed general methods for selecting two independent solutions of a differential equation $y''+I(x)y=0$ in such a way that they are most suitable for numerical and theoretical investigations. In the present paper these ideas are applied to the cases $I(x)=-(\frac{1}{4}x^2+a)$ and $I(x)=\frac{1}{4}x^2-a$. Since the character of the solutions depends essentially on the sign of $I(x)$, three intervals on the x-axis have to be distinguished, if $I(x)$ has two real roots. The author constructs fundamental systems with useful properties in all three intervals. *W. Wasow.*

***Rellich, Franz. Über Lösungen nichtlinearer Differentialgleichungen.** Festschrift zur Feier des zweihundertjährigen Bestehens der Akademie der Wissenschaften in Göttingen. I. Math.-Phys. Kl., pp. 168–174. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1951.

The first part of the paper contains an exposition of results obtained by several mathematicians confirming the author's statement that a non-linear analytic differential equation without singularities has "in general" no solutions which are entire functions. In the second part two simple criteria are derived which permit one in many cases to decide whether or not a solution of a differential equation in the real domain can be continued over a given segment of the real axis. These criteria are applied to van der Pol's equation, to the equation $y'''+2yy''=0$, and to Duffing's equation. *W. R. Wasow* (Los Angeles, Calif.).

Ważewski, Tadeusz. Sur certaines conditions de coincidence asymptotique des intégrales des deux systèmes d'équations différentielles. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. **42** (1949), 198–203 (1952). (French. Polish summary)

L'A. ripresenta la nozione di coincidenza asintotica a destra per le soluzioni di due sistemi normali di equazioni differenziali, tutte del primo ordine; e indica una condizione sufficiente perché due soluzioni siffatte coincidano asintoticamente a destra; dà poi una condizione sufficiente perchè il sistema $\dot{x}_i=f_i(t,x_1,\cdots,x_n)$, dove $i=1,\cdots,n$, possegga una (unica) soluzione che coincida asintoticamente a destra con la soluzione $y_i=x_i^{(0)}$ del sistema $\dot{y}_i=0$.

G. Scorza-Dragoni (Padova).

Mitropol'skiĭ, Yu. A. Investigation of oscillations in nonlinear systems with many degrees of freedom and slowly varying parameters. Ukrain. Mat. Žurnal **1**, no. 2, 85–98 (1949). (Russian)

Let there be given an oscillatory system with N degrees of freedom and let its kinetic and potential energies be represented by

$$T=\tfrac{1}{2}\sum a_{ij}(\tau)\dot{q}_i\dot{q}_j,\quad V=\sum b_{ij}(\tau)q_iq_j,$$

where $\tau=\epsilon t$, ϵ a small positive parameter, and $a_{ij}(\tau)$, $b_{ij}(\tau)$ are indefinitely differentiable for all $\tau\geqq 0$. Let the system be subjected to a small force with components

$$\epsilon Q_j=\epsilon Q_j^1(\tau,\theta,q,\dot{q})+\epsilon^2Q_j^2(\tau,\theta,q,\dot{q})+\cdots$$

periodic with period 2π in θ, where $\theta=\theta(\tau)$ is indefinitely differentiable for all $\tau\geqq 0$ and $d\theta/d\tau=\nu\geqq 0$. The Lagrange system is

$$\frac{d}{dt}(\sum a_{ij}\dot{q}_j)+\sum b_{ij}q_j=\epsilon Q_i. \tag{1}$$

Together with (1) introduce the auxiliary system with constant coefficients depending on both τ and ϵ:

$$\sum a_{ij}\ddot{q}_j+\sum b_{ij}q_j=0. \tag{2}$$

One assumes that (2) has N particular solutions

$$q_i^k=\varphi_i^k a\cos(\omega_k t+\alpha_k), \tag{3}$$

where φ_i^k and α_k depend on τ and ϵ. If one substitutes $\tau=\epsilon t$ in (2) and (3), (3) will only represent a solution of (2) in which frequency and amplitude vary slowly and will only be an ϵ approximation. One will now seek an asymptotic approximation to a periodic solution of (1) very near to one of the periodic solutions (3), say to the one corresponding to $k=1$. It is assumed that the only solution of (2) corresponding to equilibrium is $q=0$, and that no multiple $m\omega_1(\tau)$ is even for any τ equal to one of the $\omega_i(\tau)$, $i>1$. A solution of (1) is now looked for in the form

$$q_i=\varphi_i^1(\tau)a\cos(s\varphi+\psi)+\epsilon U_i^1(\tau,a,\theta,s\varphi+\psi)+\epsilon^2U_i^2(\tau,a,\theta,s\varphi+\psi)+\cdots,$$

where the U_i^k are periodic in θ and $s\varphi+\psi$ with period 2π and a, ψ are time functions satisfying a system

$$\dot{a}=\epsilon A_1(\tau,a,\psi)+\epsilon^2A_2(\tau,a,\psi)+\cdots,$$

$$\dot{\psi}=\omega_1(\tau)-\frac{s}{r}\nu(\tau)+\epsilon B_1(\tau,a,\psi)+\cdots.$$

Moreover $\varphi=\theta/r$ and r, s are relatively prime not too large integers which depend upon the particular resonance one wishes to consider. Everything revolves around the determination of the functions U_i^k, A_j, B_j. This is done essentially by substitution and identification of powers of ϵ, the de-

termination being so made that U_i^λ contains no zero denominator. The details are left out as too complicated to be given here. *S. Lefschetz* (Princeton, N. J.).

Atkinson, F. V. On second-order linear oscillators. Univ. Nac. Tucumán. Revista A. **8**, 71–87 (1951).

The author transforms the differential equation (1) $\omega^2(x)y''+y=0$ by the substitution $y=r\cos\theta$, $\omega y'=-r\sin\theta$ ($r>0$). The resulting equations are studied systematically. New theorems are obtained as well as new proofs of known theorems. The principal results have to do with approximations to the number of zeros of solutions of (1) and to the solutions themselves. Typical theorems are the following: Let $\omega(x)$ be positive and continuously differentiable in (a, b), and let $y(x)$ be the solution of (1) corresponding to the values $r(a)$, $\theta(a)$. Then

$$y(x)=r(a)\cos\left[\theta(a)+\int_a^x\frac{dx}{\omega}\right]+O[r(a)\exp(\operatorname*{var}_{(a,x)}\log\omega)-1].$$

If $\omega^{-2}(x)$ is replaced by $\lambda-q(x)$, where q is C^1 on $(0, \infty)$ and $q\to\infty$ as $x\to\infty$, it is shown that if there exist positive constants α, β such that $q'q^{-\alpha}>\beta$ in $(0, \infty)$, the number $n(\lambda)$ of zeros of a solution $y(x)$ in $(0, \infty)$ is given by the formula $n(\lambda)=(1/\pi)\int_0^{p(\lambda)}(\lambda-q)^{1/2}dx+O(\log\lambda)$, where $p(\lambda)$ is defined by the equation $q(p(\lambda))=\lambda$. *W. Leighton.*

Hartman, Philip. On non-oscillatory linear differential equations of second order. Amer. J. Math. **74**, 389–400 (1952).

The author continues his studies of oscillation criteria for solutions of the differential equation (1) $y''+q(x)y=0$, where $q(x)$ is real and continuous on the interval $0\leqq x\leqq\infty$. Using techniques which have been exploited by Wintner and himself, he derives necessary conditions that solutions of (1) be non-oscillatory. The principal result is the following. If

$$\sup_{0<t<\infty}\left[\left|\int_u^{u+t}q(x)dx\right|\Big/(1+t)\right]\to 0 \quad \text{as} \quad u\to\infty,$$

a necessary condition that solutions of (1) be non-oscillatory is that $\int_0^\infty q(x)dx=L$, where $-\infty\leqq L<\infty$. *W. Leighton.*

Sternberg, Robert L. Variational methods and non-oscillation theorems for systems of differential equations. Duke Math. J. **19**, 311–322 (1952).

A number of non-oscillation theorems analogous to those of Hille [Trans. Amer. Math. Soc. **64**, 234–252 (1948); these Rev. **10**, 376] and Wintner [Amer. J. Math. **69**, 87–98 (1947); Quart. Appl. Math. **6**, 183–185 (1948); these Rev. **8**, 381; **9**, 589] are proved for the differential equations associated with a fixed end point problem of Lagrange in the calculus of variations. "Non-oscillation" is taken to mean "non-existence of conjugate points", and the proofs are based on consideration of the second variation.

L. Gårding (Lund).

Taam, Choy-Tak. Oscillation theorems. Amer. J. Math. **74**, 317–324 (1952).

Let $W(x)=r(x)\exp[i\theta(x)]$ be a solution not identically zero of the differential equation

$$W''+[q_1(x)+iq_2(x)]W=0, \tag{1}$$

where $Q(x)=q_1(x)+iq_2(x)$ is an analytic function of the real variable x on $-\infty<x<\infty$. The principal theorem of the paper is the following. If $q_2(x)\not\equiv 0$, and if $y(x)$ is a real-valued solution of $y''+q_1(x)y=0$ which has b_1 and b_2 as consecutive zeros, then a non-trivial solution $W(x)$ of (1) has at most one zero on the interval $b_1\leqq x\leqq b_2$. The paper concludes with a method for constructing zero-free regions for solutions of the differential equation $W''+Q(z)W=0$, where Q is an analytic function of the complex variable z.

W. Leighton (St. Louis, Mo.).

Rasulov, M. L. Investigation of a residue method of solution of some mixed problems for differential equations. Mat. Sbornik N.S. **30**(**72**), 509–528 (1952). (Russian)

The theory of the resolvent of ordinary linear differential equations [G. D. Birkhoff, Trans. Amer. Math. Soc. **9**, 219–231 (1908)] is used to solve mixed boundary-value problems for equations of the type $M_t u=L_x u+f(x,t)$ (u and f vector-valued functions, L_x and M_t linear differential operators depending only on x and t respectively) in a region $a\leqq x\leqq b$, $t\geqq 0$. *L. Gårding* (Lund).

Miller, Kenneth S., and Schiffer, Menahem M. On the Green's functions of ordinary differential systems. Proc. Amer. Math. Soc. **3**, 433–441 (1952).

La funzione di Green G relativa a un operatore differenziale lineare ordinario di ordine n e ad n condizioni agli estremi di un intervallo (a, b) è funzione dell'operatore, delle condizioni ai limiti e dell'intervallo (a, b). Gli autori determinano le variazioni di G corrispondenti a variazioni infinitesime degli elementi da cui essa dipende.

C. Miranda (Napoli).

Titchmarsh, E. C. On the convergence of eigenfunction expansions. Quart. J. Math. Oxford Ser. (2) **3**, 139–144 (1952).

In this paper the author considers the differential equation $\varphi''+[\lambda-q(x)]\varphi=0$ subject to the boundary condition $\varphi(0)\cos\alpha+\varphi'(0)\sin\alpha=0$. It is assumed that there exist discrete eigenvalues $\lambda_0, \lambda_1, \cdots, \lambda_n\to\infty$ and corresponding eigenfunctions $\varphi_0(x), \varphi_1(x), \cdots$. The following theorem is then proved: Let $q(x)$ be twice differentiable on $0\leqq x<\infty$, and $q'(x)>0$, $q''(x)\geqq 0$, $q''(x)\leqq[q'(x)]^m$ for x sufficiently large, where $1<m<4/3$. Let $f(t)$ be in $L^2(0, \infty)$, and, in the neighborhood of $t=x$, let $f(t)$ satisfy any condition which is sufficient for the convergence of an ordinary Fourier series. Then $f(x)=\sum_0^\infty c_n\varphi_n(x)$, where $c_n=\int_0^\infty f(t)\varphi_n(t)dt$.

W. Leighton (St. Louis, Mo.).

***Burgat, Paul. Résolution de problèmes aux limites au moyen de transformations fonctionnelles.** Thesis, Université de Neuchâtel, Lausanne, 1950. 73 pp.

The paper deals with the solution of the boundary problem

$$L(y)+ry=f(x), \quad L_i(y)=\alpha_i, \quad i=1, 2, \cdots, n, \tag{1}$$

where L is a linear differential operator of the nth order, and each L_i is a linear form in the values of $y, y', \cdots, y^{[n-1]}$, at the ends of an interval (a, b). The particular problem is that of determining the "Fourier coefficients" of y with respect to the set of characteristic functions of the associated homogeneous boundary problem

$$L^*(v)+\lambda v=0, \quad V_i(v)=0, \quad i=1, 2, \cdots, n, \tag{2}$$

which is adjoint to

$$L(u)+\lambda u=0, \quad L_i(u)=0, \quad i=1, 2, \cdots, n. \tag{3}$$

It is assumed that the characteristic functions of (2) form a complete, normalizable, denumerably infinite set, and that r is a constant not one of the characteristic values of (2).

The Lagrange identity

$$\int_a^b \{\psi[L(\varphi)+r\varphi]-\varphi[L^*(\psi)+r\psi]\}dx=B(\varphi,\psi)\Big|_a^b,$$

reduces, when $\varphi=y$, and $\psi=v_k$ to

$$\int_a^b yv_k dx=\frac{1}{r-\lambda_k}\Big\{\int_a^b fv_k dx-\sum_{i=1}^{n}\alpha_i V_i(v_k).\Big\}$$

The discussion is mainly centered upon the cases in which the problem (3) is self-adjoint. The explicit formulas are calculated for a number of cases of the second order, and for some of the fourth order. *R. E. Langer.*

Myškis, A. D. Continuous dependence of the solution of a mixed problem for systems of linear differential equations upon the initial conditions and the right sides of the system. Mat. Sbornik N.S. **30(72)**, 317–328 (1952). (Russian)

The system studied is

$$A_1\partial i/\partial t+B_1\partial i/\partial x+C\partial u/\partial x+D_1 i+F_1 u=h_1,$$
$$A_2\partial u/\partial t+B_2\partial u/\partial x+C^*\partial i/\partial x+D_2 u+F_2 i=h_2,$$

with the boundary conditions

$$\Big[(C-L_1)u+M_1 i+N_1\partial i/\partial t+P_1\int_0^t i\,dt\Big]\Big|_{x=0}=0,\quad L_1^* i\Big|_{x=0}=0,$$

$$\Big[(C-L_2)u-M_2 i-N_2\partial i/\partial t-P_2\int_0^t i\,dt\Big]\Big|_{x=l}=0,\quad L_2^* i\Big|_{x=l}=0,$$

where $0\leqq x\leqq l<\infty$, $0\leqq t\leqq T<\infty$, $A_1,\cdots,P_2$ are matrices, u, i, h_1, h_2 are column matrices, the A_s, B_s, N_s, P_s are symmetric, and the $A_1,\cdots,P_2$ satisfy various continuity restrictions. The author uses a complicated integral identity to establish inequalities bounding u, i, and $\int_0^l(i^2+u^2)dx$ in terms of $u, i|_{t=0}$ and h_1, h_2. Restrictions have to be imposed on the signs of the matrices; in particular, it is sufficient that $A_s>0$, $N_s>0$, $P_s>0$, $-B_2|_{x=0}\geqq 0$, $B_2|_{x=l}\geqq 0$. Results are also given for the derivatives of u, i. Some special cases are due to Abolinya, with whom the author has written a relevant paper [Doklady Akad. Nauk SSSR **80**, 533–536 (1951); these Rev. **13**, 463]. *F. V. Atkinson* (Ibadan).

Riekstyn'š, Ē. Ya. On certain possibilities of solution of a generalized system of telegraph equations by means of the Laplace transform. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 375–381 (1952). (Russian)

The author explains at length the formation and solution of the Laplace transform of the problem

$$-\partial u/\partial x=\mathbf{R}i+\mathbf{L}\partial i/\partial t,\quad -\partial i/\partial x=\mathbf{G}u+\mathbf{C}\partial u/\partial t,$$

with fairly general boundary and initial conditions, where **R**, **L**, **G**, and **C** are constant, symmetric, positive definite square matrices. As to the more difficult task of getting back to the solution of the original problem he notes more briefly cases in which this can be done by the inverse Laplace transform. He then seeks to extend these processes to the case of parallel conductors of varying lengths and to the case of a non-homogeneous line. *F. V. Atkinson.*

Fulks, W. On the boundary values of solutions of the heat equation. Pacific J. Math. **2**, 141–145 (1952).

Let $G(x,t;s,y)$ be the well-known Green's function (expressible in terms of Jacobi's theta function ϑ_3) for the first boundary value problem of the heat equation (1) $u_{xx}-u_t=0$ in the rectangle R: $0<x<1$; $0<t<k<\infty$. Hartman and Wintner [Amer. J. Math. **72**, 367–395 (1950); these Rev. **12**, 104] gave necessary and sufficient conditions for a solution u of (1) in R to be representable in the form

$$(2)\quad u(x,t)=\int_{0+}^{1-}G(x,t;y,s)dA(y)+\int_0^t G_y(x,t;0,s)dB(s)-\int_0^t G_y(x,t;1,s)dC(s).$$

In addition, they proved (among other things) that, for the u given by (2), (3) $\lim_{s\to 0^+}u(x,t)=B'(t)$ wherever $B'(t)$ exists. The present paper improves this result by showing that the left member of (3) equals the left derivative of B wherever this derivative exists. As the author observes, this is in agreement with the intuitive fact that $u(x,t)$ should not depend on the behavior of $B(t')$ for $t'>t$.

E. H. Rothe (Ann Arbor, Mich.).

Tranter, C. J. Heat conduction in the region bounded internally by an elliptical cylinder and an analogous problem in atmospheric diffusion. Quart. J. Mech. Appl. Math. **4**, 461–465 (1951).

L'auteur étudie le problème de la conductibilité thermique dans une région limitée par un cylindre elliptique. En coordonnées elliptiques (ξ,η) cette équation est ramenée à la forme:

$$(1)\quad \frac{\partial^2 V}{\partial\xi^2}+\frac{\partial^2 V}{\partial\eta^2}=\frac{k^2}{2\alpha}(\mathrm{Ch}\,2\xi-\mathrm{Ch}\,2\eta)\frac{\partial V}{\partial t}$$

(V la température) avec les conditions aux limites: température initiale nulle, la température à la surface de l'ellipse étant constante. La solution est obtenue en utilisant la transformation de Laplace et les fonctions de Mathieu Cl_{2n} et FeK_{2n} étudiées par N. W. MacLachlan.

L'auteur donne aussi la solution limite pour un cylindre circulaire. Il montre que ce résultat est applicable dans l'étude de l'équation de la diffusion atmosphérique pour une tranche rectangulaire saturée. On est ramené par une simple transformation des variables à l'équation (1). La solution obtenue diffère de celle obtenue par D. R. Davies [même J. **3**, 51–63 (1950); ces Rev. **12**, 141]. La différence s'explique, d'après l'auteur, par l'application non justifiée du théorème d'orthogonalité. *M. Kiveliovitch* (Paris).

Ciliberto, Carlo. Su di un problema al contorno per una equazione non lineare di tipo parabolico in due variabili. Ricerche Mat. **1**, 55–77 (1952).

Let D stand for the set of points (x,y) such that $0<x<a$, $0<y<b$. Let $f_1(y)$, $f_2(y)$ be given continuous functions satisfying in $0\leqq y\leqq b$ a Hölder condition of exponent $1/2$, and also such that $f_1(0)=f_2(0)=0$. Finding a solution $u=u(x,y)$ of the following boundary value problem for the nonlinear parabolic equation (1) is under consideration

$$(1)\quad u_{xx}-u_y=f(x,y,u)\quad \text{in } D$$
$$(2)\quad u(0,y)=f_1(y),\quad u(x,0)=0,\quad u(a,y)=f_2(y),\quad \text{for } 0\leqq y\leqq b,\quad 0\leqq x\leqq a.$$

A u meeting conditions (1) and (2) is considered a solution of the boundary value problem if u is continuous in D plus its boundary and u_x, u_{xx}, u_y are continuous in D. The author, making use of a Brouwer fixed-point theorem, shows this problem has at least one solution when $f(x,y,u)$ satisfies a Hölder condition with respect to the point (x,y,u) and

$$|f(x,y,u)|\leqq g(y)|u|+m(y).$$

Here $g(y)$ and $m(y)$ are nonnegative integrable functions with $3\int_0^3 g(y)dy<1$. *F. G. Dressel* (Durham, N. C.).

Riz, P. M. Solution of the wave equation for a region near to a given one. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 345–348 (1952). (Russian)

A perturbation theory for the vibrating membrane with regard to small variations of its size is developed on the basis of a transformation of variables $\xi=x+\epsilon f_1(x, y)$, $\eta=y+\epsilon f_2(x, y)$, mapping the original membrane onto the perturbed one. The resulting nonlinear equation is treated by ordinary methods. *L. Gårding* (Lund).

Cydzik, P. V. Application of the small parameter method to the solution of problems of the characteristic vibrations of nearly rectangular plates. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 349–351 (1952). (Russian)

The method of the preceding paper is applied to a vibrating skew plate related to the rectangular one by the transformation $\xi=x$, $\eta=y-\epsilon x$. Numerical results.

L. Gårding (Lund).

Kiro, S. N. On analytic solutions of the equation

$$\frac{\partial^2 u}{\partial x_1 \partial x_2}=f\left(x_1, \cdots, x_n; u; \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1^2}, \cdots, \frac{\partial^2 u}{\partial x_n^2}, \frac{\partial^2 u}{\partial x_1 \partial x_3}, \cdots, \frac{\partial^2 u}{\partial x_{n-1} \partial x_n}\right).$$

Doklady Akad. Nauk SSSR (N.S.) **83**, 529–532 (1952). (Russian)

A proof is sketched that the equation in question (f analytic) has an analytic solution taking given analytic values when x_1 or x_2 vanish, provided that $4(\partial f/\partial u_{x_1 x_1})(\partial f/\partial u_{x_2 x_2})-1$ is not a positive number when $x_1=x_2=0$. A variant of the Cauchy-Kowalevski method due to Günther [Mat. Sbornik **32**, 26–42 (1924)] and the results of Lednev [ibid. **22(64)**, 205–266 (1948); these Rev. **10**, 253] are used.

L. Gårding (Lund).

De Donder, Th. Simplification de la méthode d'intégration d'Hadamard. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **37**, 933–935 (1951).

The author suggests the study of the finite part of the integrals used in the formulas of Hadamard by substituting in these formulas certain special functions which are known to satisfy certain special linear hyperbolic partial differential equations. *D. C. Lewis* (Baltimore, Md.).

Augé, Juan. On the classification and integration of partial differential equations of the third order. Collectanea Math. **4**, 3–53 (1951). (Spanish)

Ce mémoire se place dans la ligne des recherches de Casulleras [Collectanea Math. **1**, 3–60 (1948); ces Rev. **11**, 363] et de Teixidor [ibid. **3**, 1–71 (1950); ces Rev. **13**, 560], relatives à la résolution du problème général de Cauchy, pour certaines classes d'équations aux dérivées partielles à coefficients constants, au moyen du calcul symbolique de Fantappiè, modifié par l'emploi de l'indicatrice projective [Ann. Mat. Pura Appl. (4) **22**, 181–282 (1943); ces Rev. **8**, 589]. Dans ces recherches on réalise un rapprochement entre la géométrie algébrique et la théorie des fonctionnelles analytiques, suivant les lignes déjà tracées par Fantappiè. Les équations envisagées sont celles du type

$$\sum_{r+s+t\leq 3} a_{r,s,t}\frac{\partial^3 u}{\partial x^r \partial y^s \partial t^{3-r-s}}=f(x, y, t).$$

Pour classifier ces équations, l'auteur considère l'enveloppe constituée par les droites impropres de leurs plans caractéristiques et représentée sous la forme non homogène $\sum a_{r,s}\alpha^r\beta^s=0$ ($r+s\leqq 3$; $\alpha x+\beta y+1=0$). Un premier chapitre est consacré à l'étude des enveloppes de classe 3 et de genre 1, réduites à la forme normale $\alpha^3=4\beta^3-g_2\beta-g_3$; l'auteur les divise en 4 classes, d'après les signes de g_3 et de $\Delta\equiv g_2^3-3^3g_3^2$ (g_2, g_3 réels); ensuite il étudie la courbe ponctuelle associée, en utilisant la représentation paramétrique connue: $\alpha=\mathfrak{p}'\lambda$, $\beta=\mathfrak{p}\lambda$. Dans les autres chapitres, il s'agit essentiellement de l'application de la méthode de Fantappiè aux équations considérées, dans le cas où $\Delta>0$, $g_3<0$ (type "totalement hyperbolique"), auquel on peut encore ramener le cas $\Delta>0$, $g_3>0$. L'auteur donne une première expression de la solution dans le domaine complexe; il effectue ensuite le passage au domaine réel et il finit par présenter la solution en termes d'intégrales de volume. *J. Sebastião e Silva* (Lisbonne).

Difference Equations, Special Functional Equations

***Myškis, A. D. Lineĭnye differencial'nye uravneniya s zapazdyvayuščim argumentom. [Linear differential equations with retarded argument.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 255 pp. 8.80 rubles.

This is the first book on its subject to appear in any language. It is based largely on previous work of the author on differential equations with retarded arguments, which has been appearing since 1949 in various Russian journals. It is concerned exclusively with equations which are linear and of the first and second orders. No treatment is included, as the author points out in his preface, of inhomogeneous equations of order greater than one, of stable, unstable, or mixed equations of the second order, of orders greater than two, of singular points, or of stability criteria "in the general case." The book has five chapters. Chapter I, on general properties, begins with a statement of the problem. We shall give this here only in the case of one unknown function and for an equation of the first order. Let $-\infty<A<B\leqq\infty$. Let $f(x)$ be continuous on $[A, B)$. Let $r(x, s)$ be defined for x on $[A, B)$ and s on $[0, \infty)$, and let it satisfy the following conditions: (i) $r(x, 0)\equiv 0$; (ii) there is a continuous function of x on $[A, B)$ which for each x exceeds the total variation of $r(x, s)$ on $[0, \infty)$; (iii) there is a continuous non-negative function $\sigma(x)$ such that $r(x, s)\equiv r(x, \sigma(x))$ for $s\geqq\sigma(x)$; and (iv) $r(x, s)$ is continuous in the mean in x, that is,

$$\lim_{t\to x}\int_0^{\sigma(x)+1}|r(t, s)-r(x, s)|\,ds=0.$$

The equation is written (1) $y'(x)=\int_0^\infty y(x-s)dr(x, s)+f(x)$. Suppose that $\phi(x)$ is a continuous function defined on $(-\infty, A]$. This is the "initial function." A solution to (1) is a function $y(x)$ continuous $(-\infty, B)$, differentiable on $[A, B)$, satisfying $y(x)\equiv\phi(x)$ for x on $(-\infty, A]$ and equation (1) for x on $[A, B)$. (The author states the problem for systems of equations.) The remainder of Chapter I is concerned with existence, uniqueness, and stability questions, a comment on equations with an "advancing" argument [for this see also Myškis, Amer. Math. Soc. Translation no. **55** (1951), pp. 51–53; these Rev. **13**, 752], and a discussion of systems of equations containing derivatives of orders higher than one. Chapter II, on general properties of linear equations of the first order, contains some little-known

general properties of linear inhomogeneous equations of the first order. These results, communicated personally to the author, are due to E. G. Cvang [Zwang] and A. Ya. Lepin. Turning to homogeneous equations, the author generalizes certain results obtained in a previous paper [Mat. Sbornik 28(70), 641–658 (1951), esp. p. 651 ff.; these Rev. 13, 246] on equations of stable type. Chapter III treats linear first order homogeneous equations of unstable type. The equation (1) is unstable if $r(x, s)$ is nondecreasing in s for each x. The basic results obtained are analogous to those obtained by the author in another paper [Doklady Akad. Nauk SSSR 70, 953–956 (1950); these Rev. 11, 522] on the damping of the solutions of homogeneous equations on small semicycles [for the definition see Myškis, Mat. Sbornik 28(70), 15–54 (1951); these Rev. 13, 43]. (The author calls attention to an error in the Doklady paper: the corollary to Theorems 3 and 4 on p. 954 is incorrectly formulated.) Chapter IV is concerned with linear first order homogeneous equations of stable type. The equation (1) is stable if $r(x, s)$ is nonincreasing in s for each x. These results had been obtained previously by the author, for a simpler equation, in the first Mat. Sbornik paper referred to above and in two others [Doklady Akad. Nauk SSSR 71, 13–16 (1950); Uspehi Matem. Nauk 5, no. 2(36), 160–162 (1950); these Rev. 11, 726; 12, 106]. The author calls attention to the methods used in Chapters III and IV, the method of comparison of solutions and the method of recurrence sequences, as being particularly powerful in carrying through questions connected with the behavior of solutions, and states that equations of mixed type can be studied systematically with their aid. Chapter V studies linear second order homogeneous equations of periodic type. These equations are of the form

$$y''(x) = -\int_0^\infty y(x-s)\,dr(x, s) + f(x), \tag{2}$$

where $f(x)$, $\phi(x)$, and $r(x, s)$ are as in equation (1) and $r(x, s)$ is assumed nondecreasing in s for each x. A solution to (2) is a function $y(x)$ continuous on $(-\infty, B)$, twice differentiable on $[A, B)$, satisfying $y(x) \equiv \phi(x)$ on $(-\infty, A]$ and equation (2) for x on $[A, B)$, and having a preassigned value of $y'(A)$ (here, as in equation (1), we mean the derivative from the right). A trivial particular case of this equation is the equation $y''(x) = -My(x)$, where $M > 0$. Hence the name "periodic type." Here again the results have been carried over from results obtained previously in a simpler case [see Uspehi Matem. Nauk 4, no. 1(29), 190–193 (1949); Mat. Sbornik 28(70), 15–54 (1951); these Rev. 11, 365; 13, 43]. The book has four appendices, on "Functions of bounded variation and Stieltjes integrals," "Certain inequalities in the theory of the Stieltjes integral," "Recurrence sequences," and "The generalized linear differential equation with constant coefficients and constant lags." In the last appendix the author presents the standard Laplace transform method of solving differential-difference equations. The book is self-contained and very well organized.

J. M. Danskin (Santa Monica, Calif.).

Minorsky, Nicolas. Sur les systèmes à l'action retardée. C. R. Acad. Sci. Paris 234, 1945–1947 (1952).

In the difference-differential equation $\ddot{x} + a\dot{x} + x_h + \epsilon x_h^3 = 0$, where $x_h = x(t-h)$, let a, h, and ϵ be small parameters of the same order of magnitude. By means of his "stroboscopic" method [same C. R. 232, 2179–2180 (1951); these Rev. 13, 38] the author finds approximate expressions for the periodic solutions of this equation. He also states criteria for the stability or instability of these solutions. No stable periodic solutions exist if $h < 0$. *W. Wasow* (Los Angeles, Calif.).

Kuwagaki, Akira. Sur quelques équations fonctionnelles et leurs solutions caractéristiques. I. Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. 26, 271–277 (1951).

The equations are

$$f(x) = \lambda \sum_{i=1}^{n} f\{(x + (n-i)a + (i-1)b)/n\}, \tag{i}$$

for real values of the parameter λ, where f is to be determined throughout $[a, b]$. The method of solution is a process of iteration, at each step replacing each term on the right by a suitable linear combination obtained by using (i). The results: If $0 < \lambda < 1/n$, the only continuous solution is $f(x) \equiv 0$. If $1/n \leqq \lambda$, then either (a) $\lambda = n^{p-1}$ for a non-negative integer p, or (b) $n^{q-2} < \lambda < n^{q-1}$ for some positive integer q. In case (a), if f has a continuous pth derivative on $[a, b]$, then $f(x)$ is a polynomial of degree p, of which only the leading coefficient is arbitrary. In case (b), if f has a continuous qth derivative on $[a, b]$, then $f(x) \equiv 0$. The case $\lambda < 0$ can be reduced to the preceding cases. The characteristic solutions of (i) are the solutions corresponding to $\lambda = \lambda_p = n^{p-1}$ ($p = 0, 1, \cdots$). Starting with $f_0(x) = c$ (= arbitrary constant), $f_q(x)$ is expressed recurrently in terms of $f_{q-1}(x)$.

I. M. Sheffer (State College, Pa.).

Integral Equations

Germay, R. H. Sur les fonctions généralisant les noyaux itérés des systèmes d'équations intégrales de Volterra, de seconde espèce. Bull. Soc. Roy. Sci. Liège 21, 2–6 (1952).

In a previous note [same Bull. 16, 268–275 (1947); these Rev. 11, 38] the author generalized Volterra iterated kernels to those arising from the equation

$$\phi(x) = f_\infty(x) + \lambda \int_{a_\infty}^{x} K_\infty(x, s)\psi(s)\,ds,$$

where f_∞, K_∞ are limits of uniformly convergent sequences of given functions. The present paper extends the results to systems of integral equations of the form

$$\psi_i(x) = f_i^{(\infty)}(x) + \lambda \int_{a_\infty}^{x} \sum_{j=1}^{p} K_{ij}^{(\infty)}(x, s)\psi_j(s)\,ds \quad (j = 1, 2, \cdots, p).$$

I. A. Barnett (Cincinnati, Ohio).

Nomokonov, M. K. On the spectrum of a class of integral equations with stochastic kernel. Doklady Akad. Nauk SSSR (N.S.) 84, 445–448 (1952). (Russian)

The author proves the following result. Let $K(x, y)$ be a symmetric or symmetrizable kernel such that (i) $K(x, y) > 0$, (ii) $\int_a^b K(x, y)\,dy = 1$, (iii) $K(x, y)$ is sufficiently often differentiable with respect to x. Write

$$\Psi_0(x, y) = K(x, y),$$

$$\Psi_1(x, y) = \int_a^b \frac{\partial}{\partial x} K(x, t)\,dt,$$

$$\Psi_2(x, y) = \int_a^b \frac{\partial}{\partial x} \Psi_1(x, t)\,dt,$$

. . . . . .

$$\Psi_{n-1}(x, y) = \int_a^b \frac{\partial}{\partial x} \Psi_{n-2}(x, t)\,dt.$$

Suppose that each $\Psi_i(x, y)$ is of constant sign, and that $\Psi_i(x, b)=0$ $(1\leqq i\leqq n-1)$. Then, if the signs of the $\Psi_i(x, y)$ alternate, the first n characteristic values of $K(x, y)$ are simple and positive, and, if all $\Psi_i(x, y)$ are positive, the first n characteristic values of $K(x, y)$ are simple and alternate in sign. *F. Smithies* (Cambridge, England).

Gohberg, I. C. On an application of the theory of normed rings to singular integral equations. Uspehi Matem. Nauk (N.S.) **7**, no. 2(48), 149–156 (1952). (Russian)

The connection between normed ring theory and Mihlin's theory of singular integral equations, pointed out in general terms by the reviewer [Mat. Sbornik **28**(70), 3–14 (1951); these Rev. **13**, 46], is investigated in detail by the author for the case of the ring σ of singular integral operators A given by $A\phi=a(t)\phi(t)+b(t)S\phi(t)+T\phi$, where $\phi(t)$ is a complex-valued function of $L^2(\Gamma)$ defined on a suitable continuous contour Γ, $a(t)$ and $b(t)$ are continuous on Γ, T is a totally continuous linear operator, and S is the singular operator given by $S\phi=(1/\pi i)\int_\Gamma \phi(t)(t-\tau)^{-1}dt$, already discussed by S. G. Mihlin [Uspehi Matem. Nauk **3**, no. 3(25), 29–112 (1948); these Rev. **10**, 305]. The main result is that the non-vanishing of $a^2(t)-b^2(t)$ is a necessary condition for A to "admit regularisation", i.e., for there to be an M, bounded on $L^2(\Gamma)$, such that $MA-E$ is totally continuous, where E is the identity; the sufficiency of this condition was known. Another result is that σ is closed. Extensions are indicated for systems of integral equations.

F. V. Atkinson (Ibadan).

Finkelsztejn, L., Mikusiński, J. G.-, et Ryll-Nardzewski, C. Sur une équation intégro-différentielle. Colloquium Math. **2** (1951), 178–181 (1952).

Suppose A is the segment $0\leqq\lambda\leqq\lambda_0$ of the λ-axis, and B the segment $0\leqq t\leqq t_0$ of the t-axis. Let T be a closed region lying in the closed first quadrant of the λt-plane, and meeting the boundary of the quadrant in the two segments A and B. Consider the equation

$$\int_0^t\left[a(t-\tau)\frac{\partial x(\lambda,\tau)}{\partial\lambda}+b(t-\tau)x(\lambda,\tau)\right]d\tau=0,$$

where $a(t)$, $b(t)$ are continuous on $0\leqq t\leqq t_0$, at least one of them fails to vanish identically in the neighborhood of $t=0$, and $x(\lambda, t)$, $\partial x(\lambda, t)/\partial\lambda$ are continuous in T. The theorem of the paper is that, if T is the right triangle with A and B as two sides, if $x(\lambda, t)=0$ on B, and if the equation is satisfied in T, then $x(\lambda, t)=0$ throughout T. An example shows that the theorem is false if T is the rectangle with adjacent edges, A and B. One cannot infer that $x(\lambda, t)$ vanishes except in the triangle. The work is related to an earlier paper by Mikusiński on differential equations in abstract spaces [Ann. Soc. Polon. Math. **22**, 157–160 (1949); these Rev. **12**, 8]. *A. E. Taylor* (Los Angeles, Calif.).

Nickel, K. Zusatz zu J. Dörr: Strenge Lösung der Integralgleichung für ein Flügelgitter. Ing.-Arch. **20**, 6–7 (1952).

An explicit formula is given for the value of a constant k which appeared in the solution $\Gamma(\xi)$ of an integral equation discussed by J. Dörr [Ing.-Arch. **19**, 66–68 (1951); these Rev. **13**, 44], under the assumption that $\Gamma(1)=0$. Also determined explicitly is a formula for $\int_{-1}^{1}\Gamma(\xi)d\xi$.

E. Reissner (Cambridge, Mass.).

Dörr, J. Über zwei mit der Tragflügeltheorie in Zusammenhang stehende Integralgleichungen. Ing.-Arch. **20**, 88–90 (1952).

In a previous paper [Ing.-Arch. **19**, 66–68 (1951); these Rev. **13**, 44] the author has derived the explicit solution of the integral equation

$$f(x)=\frac{1}{2h}\int_{-1}^{1}g(y)\cot\frac{\pi(x-y)}{h}dy$$

under the assumption that $h=h'+ih''$ and either $h''=0$ or $h'=0$, $-1>h''>+1$. In the present note he shows that the solution is actually valid for all real h' and h''.

E. Reissner (Cambridge, Mass.).

Functional Analysis, Ergodic Theory

***Lyusternik, L. A., i Sobolev, V. I. Elementy funkcional'nogo analiza. [The elements of functional analysis.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 360 pp. 11.30 rubles.

This book has been developed on the basis of an article by the first-named author [Uspehi Matem. Nauk **1**, 77–140 (1936)] and lectures by the second. It forms a clearly written and very good introduction to the parts of Banach space theory most frequently used in analysis. The standpoint throughout is analytical: numerous discussions of particular spaces serve as illustrations of the general theory. The following is a brief account of the contents with particular reference to less familiar topics treated.

Chap. I, "Metric spaces". The general theory, with the most familiar linear spaces as examples. Contractions are discussed and their theory applied to iterative solutions of equations. Compactness is discussed with reference to criteria for compact sets in L_p and l_p and almost periodic functions. Chap. II, "Linear spaces and linear operators". Main properties (boundedness, inverse) of operators on Banach spaces; finite-dimensional spaces. Chap. III, "Linear functionals". Hahn-Banach Theorem. Forms of linear functionals. Properties of conjugate spaces, including weak compactness of unit sphere (for the case when the original spaces is separable). Criteria for weak convergence. Imbedding in $C(0, 1)$. Chap. IV, "Completely continuous operators". The Riesz-Schauder theory for spaces with a basis. Chap. V, "Spectral theory of self-adjoint operators". The theory of bounded self-adjoint operators in a separable Hilbert space. Chap. VI, "Some problems of nonlinear analysis". Derivatives of vector-valued functions both of real and of vector arguments and the Riemann integral for a vector function of a real argument are defined. The integral is used to discuss the existence of solutions of equations $dx/dt=f(x, t)$ by the Picard iteration process. Multilinear forms, and hence polynomial functions, of vectors are defined: and derivatives of vector functions of vectors of all orders are defined by means of polynomial approximations to differences: further, a "strong" (Fréchet) and a "weak" (strong limit of $t^{-1}[f(x+th)-f(x)]$) differential coefficient are defined, and equivalence of the various definitions discussed. Functions of several variables are treated. A theorem on implicit functions with applications is given. Tangents to manifolds $\varphi(x)=0$ and extremal problems are discussed. Two appendices discuss the Hölder and Minkowski inequalities, and the equivalence of the alternative definitions

of differentiation for ordinary functions. A bibliography lists the main works available in Russian.

The book is accurately written, but there are occasional oversights. No distinction has been made between the cases $p=1$ and $p>1$ in discussing functionals and reflexivity on L_p and l_p. In the last chapter conditions for proofs and theorems are not always given fully. *J. L. B. Cooper.*

Albrycht, J. L'Hôpital's rule for vector-valued functions. Colloquium Math. **2** (1951), 176–177 (1952).

Let X be a real Banach space, $x(t)$ a function on $a<t\leqq b$ to X, weakly differentiable on $a<t<b$ and weakly continuous at b, with $x(b)=0$. Let $\gamma(t)$ be a real-valued function differentiable on $a<t<b$, continuous at b, with $\gamma(b)=0$. If $y(t)$ is the weak derivative of $x(t)$, then as $t\to b$,

$$\lim x(t)/\gamma(t)=\lim y(t)/\gamma'(t),$$

provided the limit on the right exists (these latter limits in the norm topology). The proof uses some facts about convex sets in X. *A. E. Taylor* (Los Angeles, Calif.).

Kračkovskiĭ, S. N., and Vinogradov, A. A. On a criterion of uniform convexity of a space of type B. Uspehi Matem. Nauk (N.S.) **7**, no. 3(49), 131–134 (1952). (Russian)

Clarkson [Trans. Amer. Math. Soc. **40**, 396–414 (1936)] called a normed space B uniformly convex if for each ϵ, $0<\epsilon\leqq 2$, there is a $\delta(\epsilon)>0$ such that $\|x+y\|\leqq 2(1-\delta(\epsilon))$ whenever $\|x\|=\|y\|=1$ and $\|x-y\|\geqq\epsilon$. This note proves the following theorem: Whenever $\|x\|=\|y\|=1$, let H_x and H_y be hyperplanes of support of the unit sphere at x and at y, respectively; let M_{xy} be the common part of H_x and H_y. Then B is uniformly convex if and only if for each ϵ, $0<\epsilon\leqq 2$, there exists $\delta'(\epsilon)>0$ such that $\|z\|\geqq 1+\delta'(\epsilon)$ whenever $\|x-y\|\geqq\epsilon$ and $z\in M_{xy}$. The proof is an unnecessarily complicated contradiction argument which shows only existence; the authors do not give the simple direct calculations which would show $\delta'(\epsilon)\geqq\delta(\epsilon/2)$ and $\delta(\epsilon)\geqq\delta'(\epsilon/4)$. *M. M. Day.*

Vaĭnberg, M. M. On invariant directions of certain completely continuous operators. Doklady Akad. Nauk SSSR (N.S.) **83**, 785–788 (1952). (Russian)

This note extends results of Rutman [Kreĭn and Rutman, Uspehi Matem. Nauk **3**, no. 1(23), 3–95 (1948), §9; these Rev. **10**, 256] on fixed directions of not-necessarily-linear, completely continuous operators. In order to avoid the use of cones this proof requires that E be an infinite-dimensional Banach space. Theorem 1. Let F be completely continuous and defined on the boundary ω' of a convex open set ω containing 0 in E, and suppose that $\|F(x)\|\geqq a>0$ for each $x\in\omega'$. Then there exists at least one x' in ω' and a $\lambda>0$ such that $F(x')=\lambda x'$. Theorem 2 weakens the last hypothesis needed in Theorem 1. An integral equation is displayed as an application. *M. M. Day* (Urbana, Ill.).

Vaĭnberg, M. M. On the differential and gradient of functionals. Uspehi Matem. Nauk (N.S.) **7**, no. 3(49), 139–143 (1952). (Russian)

Let t be a real variable and $f(x)$ a functional defined on the real Banach space E. The following two theorems are proved. (1) If the Gateaux differential $Df(x,h)$ defined by

$$\lim_{t\to 0}\frac{f(x+th)-f(x)}{t}=Df(x,h),\quad x,h\in E,$$

exists and is a continuous function of x for all x in some open set U of E and of h for all sufficiently small h, then $Df(x,h)$ is linear in h, hence of the form $(F(x),h)$ where $F(x)=\operatorname{grad} f(x)$ is a function from $U\subset E$ to the conjugate space E^*. (2) If $F(x)$ is continuous on U, then $Df(x,h)$ coincides with the Fréchet differential $df(x,h)$ defined by

$$f(x+h)-f(x)=df(x,h)+\omega(x,h),\quad \lim_{\|h\|\to 0}\frac{\omega(x,h)}{\|h\|}=0.$$

M. Golomb (Lafayette, Ind.).

Vaĭnberg, M. M. On some variational principles in the theory of operator equations. Uspehi Matem. Nauk (N.S.) **7**, no. 2(48), 197–200 (1952). (Russian)

The author observes that a theorem of Lyusternik [Mat. Sbornik **41**, 390–401 (1934)] implies the following result of Golomb [Math. Z. **39**, 45–75 (1934)]: If in a real Hilbert space H there is given an operator $F(x)=\operatorname{grad} f(x)$ and also a completely continuous self-adjoint operator Ax; then the equation $\mu x=A^2F(x)$ has at least two different solutions in each sphere $\|x\|\leqq r$, corresponding to real eigen-values μ. He also deduces connections with the work of Rothe [Ann. of Math. **49**, 265–278 (1948); these Rev. **10**, 461].

J. M. Danskin (Santa Monica, Calif.).

Krasnosel'skiĭ, M. A. On the estimation of the number of critical points of functionals. Uspehi Matem. Nauk (N.S.) **7**, no. 2(48), 157–164 (1952). (Russian)

Let there be given on a connected manifold K a continuous involution A without invariant points, i.e., a continuous operator on K to K for which $Ax\neq x$ and $A^2x=x$. A set of the first kind on K is a compact set none of whose components contains both x and Ax. A set of kind n on K is a set each of whose compact parts may be divided into n sets of the first kind and for which some compact part may not be divided into $(n-1)$ sets of the first kind. This concept is closely related to that of category. The author illustrates its use to obtain a theorem of Lyusternik on the number of critical points of an even nonnegative weakly continuous functional on the sphere S of Hilbert space, and related results, without reference to the properties of projective space. He generalizes this concept to manifolds on which there are given periodic transformations of period r.

J. M. Danskin (Santa Monica, Calif.).

Wermer, John. On invariant subspaces of normal operators. Proc. Amer. Math. Soc. **3**, 270–277 (1952).

The paper is in two parts. In the first part T is assumed to be a bounded normal operator on a Hilbert space H, with distinct eigenvalues $\lambda_1,\lambda_2,\cdots$, and a set of eigenvectors spanning H. There is said to be spectral synthesis for T if every closed subspace which is invariant under T is spanned by a set of eigenvalues of T. It is then proved that each one of the following three conditions is equivalent to spectral synthesis for T: (i) every closed invariant subspace (not (0)) contains at least one eigenvector; (ii) every closed subspace invariant under T is also invariant under T^*; (iii) if $\sum_{i=1}^{\infty}|\omega_i|<\infty$ and $\sum_{i=1}^{\infty}\lambda_i^n\omega_i=0$ for $n=0,1,2,\cdots$, then $\omega_1=\omega_2=\cdots=0$. The issue of spectral synthesis is related to the uniqueness problem for Borel series $\sum_{i=1}^{\infty}A_i/(z-\alpha_i)$. This connection is exploited to show that spectral synthesis can fail if $0<|\lambda_1|<|\lambda_2|<\cdots<r$, but that there will be spectral synthesis if the set $\{\lambda_i\}$ lies on a Jordan curve. The function-theoretic proof of this last fact depends on a 1950 Uppsala thesis not available to the reviewer. It is also shown that there is spectral synthesis if there is no infinite subsequence $\{\lambda_{n_i}\}$ such that $|\lambda_{n_1}|<|\lambda_{n_2}|<\cdots$. Still another sufficient (but not necessary) condition for spectral synthesis

is that there exist a sequence of polynomials $\{P_n\}$ such that $P_n(T)$ converges weakly to T^*.

In the second part of the paper T is assumed to be bounded and normal, but nothing is assumed about its eigenvectors. For T to have property (P) means that every closed subspace invariant under T is also invariant under T^*. A sufficient condition for (P) is that the spectrum of T (hereafter denoted by $\sigma(T)$) not separate the plane and have no interior. In the spectral representation for T, let $E(S)$ denote the projection operator associated with the Borel set S. Let m be a nonnegative measure defined on Borel sets. The notation $m \vdash T$ means that $E(S)=0$ implies $m(S)=0$. First assume there is a circular disk d contained in $\sigma(T)$, and let $m(S)$ be the Lebesgue measure of $S \cap d$. Then $m \vdash T$ implies that T does not have property (P). A similar theorem, with the condition both necessary and sufficient, is given for the case in which $\sigma(T)$ lies on a closed rectifiable Jordan curve bounding a simply connected region.

A. E. Taylor (Los Angeles, Calif.).

Chiang, Tse-Pei. A theorem on the normalcy of completely continuous operators. Acta Sci. Math. Szeged **14**, 188–196 (1952).

The main result of the paper is as follows. Let A be a completely continuous linear operator in Hilbert space, with eigen-values (α_n) and inverse singular values (σ_n), both sequences being arranged in decreasing order of absolute magnitude. Let $\psi(x)$ be a strictly increasing convex function defined for $x \geqq 0$. Then a sufficient condition for the operator A to be normal is that

$$\liminf_{n\to\infty} \sum_{i=1}^{n} \{\psi(\sigma_i^2) - \psi(|\alpha_i|^2)\} = 0.$$

The author gives a counterexample showing that the result ceases to be true if it is only supposed that $\psi(x)$ is strictly increasing for $x \geqq 0$ and $\psi(e^t)$ is a convex function of ξ.

F. Smithies (Cambridge, England).

Kreĭn, M. G. On an application of the fixed point principle in the theory of linear transformations of spaces with an indefinite metric. Uspehi Matem. Nauk (N.S.) **5**, no. 2(36), 180–190 (1950). (Russian)

The following theorem is proved. Let

$$J(x, y) = \sum_{j,k=1}^{n} g_{jk}\xi_j\bar{\eta}_k$$

be an Hermitian form which, upon transformation by SU, where U is unitary and S is a diagonal matrix, assumes the form $\sum_{j=1}^{p}\xi_j\bar{\eta}_j - \sum_{j=p+1}^{n}\xi_j\bar{\eta}_j$. Let A be a linear transformation such that $J(Ax, Ax) \geqq J(x, x)$ whenever $J(x, x) \geqq 0$. Then there exists a p-dimensional A-invariant subspace in which $J(x, x)$ is non-negative and in which all characteristic roots of A have absolute value $\geqq 1$. This assertion, under the stronger hypothesis $J(Ax, Ax) = J(x, x)$ for all x, is due to Frobenius [J. Reine Angew. Math. **84**, 1–63) (1877)]. This weaker theorem was extended to l_2, $(J(x, x) = \sum_{j=1}^{p}\xi_j\bar{\xi}_j - \sum_{j=p+1}^{n}\xi_j\bar{\xi}_j)$ by Pontryagin [Izvestiya Akad. Nauk SSSR. Ser. Mat. **8**, 243–280 (1944); these Rev. **6**, 273]. (A similar extension is given here for the stronger theorem.) The proof, which is brief, depends upon an ingenious application of Brouwer's fix-point theorem.

E. Hewitt (Seattle, Wash.).

Kreĭn, M. G. On Hermitian operators with directed functionals. Akad. Nauk Ukrain. RSR. Zbirnik Prac' Inst. Mat. **1948**, no. 10, 83–106 (1948). (Ukrainian. Russian summary)

A detailed discussion of and diverse applications for a theorem announced earlier [C. R. (Doklady) Acad. Sci. URSS **53**, 3–6 (1946); these Rev. **8**, 277]. The main theorem [cf. the review cited] is proved for the case that $\mathfrak{H}$ is an inner product space in which $(f, f)=0$ is possible for $f \neq 0$; one notes also that the matrix-function $T(\lambda) = \|\tau_{jk}(\lambda)\|_{j,k=1}^{p}$ is Hermitian for every λ, that $T(0)=0$, and that $T(\lambda-0)=T(\lambda)$ for every λ. The matrix function which produces the inner product in $\mathfrak{H}$ is unique if and only if at least one of the defect-numbers of the operator A is zero. Of the applications, we here cite the following. Let $\mathfrak{L}$ be the linear space of all p-dimensional vector-functions $g(t) = \{g_1(t), \cdots, g_p(t)\}$ on $[0, a]$ such that $g(0) = g(a)$, and let $A(g) = i\, dg/dt$, for all $g \in \mathfrak{D}(A)$ = the set of g's such that $A(g)$ exists and is in $\mathfrak{L}$. Plainly, the functions $\Phi_j(g_j\lambda) = \int_0^a e^{\Omega\lambda t} g_j(t)\,dt$ satisfy the hypotheses of the main theorem. Let $F(t) = \|f_{jk}(t)\|$, $j, k = 1, \cdots, p$, $|t| \leqq a$, be a continuous matrix-function. Writing $(g, f) = \int_0^a \int_0^a \sum_{j,k=1}^{p} f_{jk}(s-t) g_k(s) \overline{f_j(t)}\,ds\,dt$, we see that $\mathfrak{L}$ is an inner product space if and only if

$$\sum_{j,k=1}^{n} (x^{(j)})^* F(s_j - s_k) x^{(k)} \geqq 0$$

for all $s_1, \cdots, s_n \in [0, a]$ and $x^{(1)}, \cdots, x^{(n)}$ p-dimensional complex vectors. Call the class of all such matrix-functions $\mathfrak{P}_a$. Then the main theorem, applied to this case, asserts that $F(s) = \int_{-a}^{a} e^{\Omega s}\, dT(\lambda)$ $(|s| \leqq a)$, for a matrix $T(\lambda)$ as described above, if and only if $F \in \mathfrak{P}_a$.

E. Hewitt.

Kreĭn, M. G. The fundamental propositions of the theory of representations of Hermitian operators with deficiency index (m, m). Ukrain. Mat. Žurnal **1**, no. 2, 3–66 (1949). (Russian)

Let A be a closed Hermitian operator with equal finite deficiency indices (m, m) in a Hilbert space H. Let M_z be the range of $A - zI$; A is called simple if the M_z, z not real, have no point but 0 in common. Only such A are considered in this paper.

§2 shows how to choose analytic bases, $\varphi_1(z), \cdots, \varphi_m(z)$, for the orthogonal complements N_z of the M_z. Letting $u_1, \cdots, u_n$ be a basis for a fixed subspace M of H, $M_z \cap M = (0)$ if and only if $\Delta_u(z)$, the determinant of $(u_j, \varphi_k(z))$, is not zero; M can be chosen so that Δ_u is not identically zero in either half plane; it is holomorphic in the resolvent set of A. Then for all z which are not zeros of Δ_u a projection of H on M along M_z defines $f_M(z) = \sum_j \varepsilon_{uj} f_u^j(z) u_j$. Theorem 1. The mapping $f \to f_M$ is an isomorphism of H with a space H_M of M-valued functions of z, each meromorphic in the upper and the lower half plane. A is carried by this isomorphism into the operation of multiplication by z.

In §3 the basis u is called quasiregular if the set R, of all $f \in H$ for which f_M is holomorphic along the real axis, is dense in H. This section begins the study of the relations between spectral functions associated with A and matrix-valued functions defined on the real axis. §4 contains results on bounds for integrals of functions of class (N); that is, holomorphic functions such that $\log^+ f(z)$ has a harmonic majorant. The results are applied in §5 to estimate bounds of $\|f_M(z)\|/\|f\|$ in neighborhoods where all f_M are regular. In §6 it is shown that if a is non-real, then a has a neighborhood in which all f_M are holomorphic if and only if $\Delta_u(a) \neq 0$.

§7 introduces the function $\nabla_M(z) = \sup_{\|f\|=1} \|f_M(z)\|$. In §8, A is called entire if M can be chosen so that each f_M is

an entire function of z. Then ∇_M is of $\leqq$exponential type and an entire operator can be classified as either minimal, if $\limsup_{|z|\to\infty} \log \nabla_M(z)/|z| = 0$, or normal, if the lim sup is finite but non-zero.

§9 is an aside discussing related work of the author [see the preceding review] in case A has a family of functionals $\Phi_j(f;\lambda)$ satisfying a number of conditions. §10 gives examples of both types of entire operators. A matrix moment problem is shown to lead to a minimal entire operator; the corresponding continuous problem, of representing matrix-valued, Hermitian-positive functions, leads to entire operators of normal type.

The reviewer was never able to find the proof, promised in the introduction, that for each A the subspace M can be chosen so that all the functions f_M are holomorphic in the upper and the lower half plane. *M. M. Day.*

Krasnosel'skiĭ, M. A., and Rutickiĭ, Ya. B. Linear integral operators in Orlicz spaces. Doklady Akad. Nauk SSSR (N.S.) **85**, 33–36 (1952). (Russian)

For definitions and notations, the reader is referred to the review of Krasnosel'skiĭ and Rutickiĭ, same Doklady **81**, 497–500 (1951); these Rev. **13**, 357. The authors investigate operators of the form: $Au(x) = \int_G K(x,y)u(y)dy$, where G is a compact subset of n-dimensional Euclidean space, $K(x,y)$ is measurable on $G\times G$, and the domain and range of A are respectively in Orlicz spaces $L_{M_1}(G)$, $L_{M_2}(G)$. Here M_1 and M_2 are N'-functions, N_1 and N_2 their complementary functions. Conditions which insure that A is continuous (completely continuous) are given without proof. The results are: 1) Let Φ be an N'-function such that if $u\in L^*_{M_1}(G)$, $v\in L^*_{N_2}(G)$, then $u(x)v(y)\in L^*_\Phi(G\times G)$ and $\|uv\|_\Phi \leqq C\|u\|_{M_1}\|v\|_{N_2}$. Then if $K(x,y)\in L^*_\Psi(G\times G)$ (Ψ the function complementary to Φ), the operator A is a continuous mapping from $L^*_{M_1}(G)$ into $L^*_{M_2}(G)$. 2) If $\Psi = M_2(N_1)$ or $\Psi = N_1(M_2)$, and if $K\in L^*_\Psi(G\times G)$, then A is a continuous mapping from $L^*_{M_1}(G)$ into $L^*_{M_2}(G)$. If an N'-function M is such that for some u_0 and some Q, $|M(uv)| \leqq QM(u)M(v)$ for $u, v \geqq u_0$, M is said to satisfy the Δ'-condition. If $L^*_{M_1}(G) \supset L^*_M(G)$, then one writes $M > M_1$. In terms of these notions one has: 3) Let Ψ be an N'-function satisfying the Δ'-condition and $\Psi > M_2$, $\Psi > N_1$, and $K\in L^*_\Psi(G\times G)$; then the conclusion of 1) and 2) is valid.

A function u is said to have an absolutely continuous norm in $L^*_M(G)$ if for $\epsilon > 0$ there is a $\delta > 0$ such that $\|u\chi_E\|_M < \epsilon$ if measure $E = m(E) < \delta$, where χ_E is the characteristic function of E. 4) If K has an absolutely continuous norm in $L^*_\Phi(G\times G)$, then under the hypotheses of 1), 2), or 3) A is a completely continuous mapping of $L^*_{M_1}(G)$ into $L^*_{M_2}(G)$. 5) Conclusion of 4) holds if Ψ satisfies the Δ_2-condition [see references]. The paper concludes with two theorems which give conditions for the compactness of a subset m of $L^*_M(G)$. *B. R. Gelbaum.*

Fullerton, R. E. A characterization of L spaces. Fund. Math. **38**, 127–136 (1951).

The following characterization of Banach spaces X which are equivalent to Lebesgue spaces $L(M, m)$ where $m(M) = 1$ is proved. (1) there lies on the unit sphere a maximal convex set F such that the cone C of all non-negative multiples of elements of F has the Clarkson [Ann. of Math. **48**, 845–850 (1947); these Rev. **9**, 192] property: given x and y there is a z such that $z+C = (x+C)\cap(y+C)$; (2) the unit ball is the convex hull of $F\cup(-F)$; (3) there is an element e in F such that given x in C there is a y in C such that both x and e lie in $y+C$. The proof of the nontrivial part of this theorem is carried out by showing that the ordering based on this cone leads to an abstract (L) space in the sense of Kakutani, and by then applying Kakutani's theorem [ibid. **42**, 523–537 (1941); these Rev. **2**, 318]. *R. Arens.*

Berman, D. L. On a class of linear operators. Doklady Akad. Nauk SSSR (N.S.) **85**, 13–16 (1952). (Russian)

Let $\bar{C}$ be the space of all continuous functions $f(x)$ of period 2π, with norm $\|f\|_{\bar{C}} = \max|f(x)|$. Let us write $f(x+t) = f_t(x)$. Marcinkiewicz [Acta Litt. Sci. Szeged **8**, 127–130 (1937)] has proved the following fact. Let n be fixed and let $I(f,x)$ denote the Lagrange interpolation polynomial of order n coinciding with f at $2n+1$ prescribed points. Then

$$\frac{1}{2\pi}\int_0^{2\pi} I(f_t, x-t)dt = S_n(f,x),$$

where $S_n(f,x)$ denotes the nth partial sum of the Fourier series of f. The author generalizes this as follows. Let $U(f,x)$ be a linear operator from $\bar{C}$ to $\bar{C}$ and such that (a) $U(f,x)$ is always a trigonometric polynomial of order $\leqq n$; (b) for any trigonometric polynomial T of order $\leqq n$ we have

$$U(T,x) = \int_0^{2\pi} T(x+t)\Phi(t)dt$$

where $\Phi(t)$ is a fixed trigonometric polynomial of order n. Under these assumptions,

$$\frac{1}{2\pi}\int_0^{2\pi} U(f_t, x-t)dt = \int_0^{2\pi} f(x+t)\Phi(t)dt.$$

The proof is the same as that of the special case mentioned above. *A. Zygmund* (Chicago, Ill.).

*Giorgi, Giovanni. **Progressi e applicazioni del calcolo operatorio funzionale.** Reale Accademia d'Italia, Fondazione Alessandro Volta, Atti dei Convegni, v. **9** (1939), pp. 281–289, Rome, 1943.

Der Autor skizziert eingangs verschiedene Richtungen der modernen Analysis. Dann wird der von ihm aufgestellte Operatorencalcul [Rend. Sem. Mat. Fis. Milano **8**, 189–214 (1934)] diskutiert und mit den entsprechenden Arbeiten von Wiener und Fantappiè verglichen. Abschliessend wird auf die Nützlichkeit der Theorie zur Lösung physikalischer Probleme und deren Vorteile gegenüber dem Heaviside-Calcul und den Laplacetransformationen hingewiesen.
H. G. Haefeli (Cambridge, Mass.).

Succi, Francesco. Su i funzionali analitici invarianti di prolungamento delle funzioni analitiche. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **9**, 79–103 (1950).

Eine abhängige Veränderliche F wird ein invariantes Funktional der analytischen Fortsetzung genannt, wenn sie von einer analytischen Funktion in einer solchen Weise abhängt, dass sie durch ein Funktionselement bestimmt ist und sich beim Übergang zu einem andern nicht ändert. Für ein solches Funktional gilt daher:

$$F[f(z)] = F[f(z+h)].$$

Da ein Funktionselement eine lokal-analytische Funktion ist, wird nach den analytischen, invarianten Funktionalen der Fortsetzung gefragt. Die Ableitung eines solchen ist vom Typus des geschlossenen Zykels:

$$F'[f(z);\alpha] = F'[f(z+h);\alpha-h].$$

Die gesuchten Funktionale sind daher in der Klasse der primitiven Funktionale derjenigen des geschlossenen Zykels zu finden. Nun zeigt der Autor, dass es in dieser Klasse keine nicht konstante Funktionale gibt, die für Funktionen definiert sind, welche im Unendlichen singulär (nicht definiert) sind. Damit gibt es auch keine nicht konstante, analytische, invariante Funktionale der Fortsetzung im strengen Sinn. Der Beweis stützt sich auf die von Fantappiè gefundene Reihenentwicklung der Funktionale des geschlossenen Zykels und einen Satz über die Singularitäten der Funktionen

$$v_n(\alpha_1, \alpha_2, \cdots, \alpha_n) = F^{(n)}[k; \alpha_1, \alpha_2, \cdots, \alpha_n].$$

Der Verfasser beweist, dass die v_n für ein beliebiges analytisches Funktional, das für eine Konstante $k \neq 0$ definiert ist, nie in uneigentlichen Punkten ausserhalb der Koordinatenhyperebenen singulär werden können, wenn die $\alpha_1, \alpha_2, \cdots, \alpha_n$ in einem projektiven S_n gedeutet werden.

Die analytischen, invarianten Funktionale der Fortsetzung, die existieren, sind somit nur definiert für Funktionen, welche im Unendlichen definiert sind (nicht für beliebige Funktionselemente), und werden daher mit Vorteil mit Hilfe der Laurententwicklung um Unendlich berechnet. Anschliessend wird ein von Calugareano [Mathematica, Cluj **12**, 164–179 (1936)] gefundenes, invariantes Funktional der Fortsetzung diskutiert und seine Entwicklung in eine Reihe von Fantappiè gegeben. Abschliessend wird bezeigt, dass jedes lineare, primitive Funktional derjenigen des geschlossenen Zykels eine Invariante der analytischen Fortsetzung ist und eine konstante Indikatrix besitzt.

H. G. Haefeli (Cambridge, Mass.).

Pucci, Carlo. Sulla continuità dei funzionali analitici. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **10**, 290–296 (1951).

Il s'agit d'une contribution à l'étude des fonctionnelles analytiques de Fantappiè. Haefeli et Pellegrino [Comment. Math. Helv. **21**, 225–246 (1948); **23**, 153–173 (1949); ces Rev. **9**, 515; **11**, 442] ont démontré que toute fonctionnelle analytique localement bornée est continue. L'auteur présente une démonstration plus directe de ce fait, donnée pour les fonctionnelles de fonctions de plusieurs variables; en outre, il démontre que les points de continuité d'une fonctionnelle analytique $F[y]$ forment un ensemble ouvert et il donne une propriété des points hypothétiques de discontinuité de la fonctionnelle $F[y]$. Seulement, le rapporteur doit signaler que, contrairement à ce que pense l'auteur, il a déjà été établi que toute fonctionnelle analytique (au sens de Fantappiè) est continue [Portugaliae Math. **9**, 1–130 (1950); ces Rev. **11**, 524]. *J. Sebastião e Silva.*

Kaplansky, Irving. Symmetry of Banach algebras. Proc. Amer. Math. Soc. **3**, 396–399 (1952).

A complex Banach algebra is called a $*$-algebra if it possesses an involution $*$ (i.e., a conjugate linear anti-automorphism of period two). A $*$-algebra A is called symmetric if x^*x is quasiregular for every x in A; it is called C-symmetric if every closed, commutative $*$-subalgebra is symmetric. Gelfand and Neumark [Mat. Sbornik **12(54)**, 197–213 (1943); these Rev. **5**, 147] proved that every B^*-algebra (i.e., $*$-algebra A such that $\|xx^*\| = \|x\|^2$ for every x in A) is C-symmetric, and conjectured that every B_*-algebra is, in fact, symmetric. In partial support of the stronger conjecture that every C-symmetric algebra is symmetric, the author proves the following theorem (generalizing a previous result of his [Ann. of Math. **53**, 235–249 (1951); these Rev. **13**, 48] in support of the original Gelfand-Neumark conjecture): If A is a C-symmetric algebra with continuous involution, then for no element x of A can $-x^*x$ be a nonzero idempotent. The proof is based on the following purely algebraic lemma, which seems to have considerable independent interest: Let F be a field, and A an algebra over F which is generated by elements x and y satisfying the relations $x^2 = \alpha x$, $y^2 = \beta y$, where α and β are in F. Then A satisfies the polynomial identity $\sum \pm x_1x_2x_3x_4 = 0$, where the sum is taken over all permutations, and the sign is determined by the parity of the permutation.

E. Michael (Chicago, Ill.).

Kaplansky, Irving. Group algebras in the large. Tôhoku Math. J. (2) **3**, 249–256 (1951).

Let $W(G)$ be the weakly closed operator algebra generated by the regular representation of a locally compact group G. Mautner has recently studied the question of determining the type of the factors into which $W(G)$ may be decomposed. In particular, when G is discrete, he has given two sufficient conditions that almost every factor in the decomposition be of type II_1 [Duke Math. J. **17**, 437–441 (1950); Trans. Amer. Math. Soc. **70**, 531–548 (1951); these Rev. **12**, 588; **13**, 11]. On the other hand, recent work of Dixmier and the author has led to a notion of type for general weakly closed operator algebras. This notion permits the formulation of Mautner's problem without considering direct integral decompositions and in the present paper the author obtains such global versions of the two theorems of Mautner referred to above. In one case the proof becomes very much simpler and shorter. Other results include the following. Let G be discrete and infinite and let its center be cyclic of prime order n and coincide with the commutator subgroup of G. Then $W(G)$ is the direct sum of $n-1$ factors of type II_1 and a commutative algebra. Any $*$ representation of a GCR algebra (as defined by the author [Trans. Amer. Math. Soc. **70**, 219–255 (1951); these Rev. **13**, 48]) is of type I. Any unitary representation of a GCR group is of type I.

G. W. Mackey (Cambridge, Mass.).

Misonou, Yosinao, and Nakamura, Masahiro. Centering of an operator algebra. Tôhoku Math. J. (2) **3**, 243–248 (1951).

The authors state and give a proof of a result giving a necessary and sufficient condition that a C^*-algebra of a special type admit a "centering" c, this being defined to be like the center-valued trace on finite W^*-algebras treated by Dixmier, except that the condition $c(c(A)B) = c(A)c(B)$ is replaced by the condition that $\omega(A) = \omega(c(A))$ for all traces ω (= states such that $\omega(AB) = \omega(BA)$). *I. E. Segal.*

Umegaki, Hisaharu. On some representation theorems in an operator algebra. II. Proc. Japan Acad. **27**, 501–505 (1951).

The author states and sketches proofs of results to the effect that (1) any continuous unitary representation of a separable locally compact group is the direct integral (in the sense of von Neumann) of irreducible such representations, and (2) the analogous result for 2-sided representations. Similar results have been presented by other authors [see Segal, Mem. Amer. Math. Soc., no. **9** (1951); and re (1) see also Mautner, Ann. of Math. **52**, 528–556 (1950); and Godement, ibid. **53**, 68–124 (1951); these Rev. **13**, 472; **12**, 157, 421]. There is an apparent gap in the author's outline of the proof of (2), in that the invariance under the basic conjugation of the abelian algebra relative to which the decomposition is made is not treated. *I. E. Segal.*

Umegaki, Hisaharu. On some representation theorems in an operator algebra. III. Proc. Japan Acad. **28**, 29–31 (1952).

The author states without proof extensions of some of his earlier work on traces [same Proc. **27**, 328–333 (1951); these Rev. **13**, 756; and the paper reviewed above] and of work on motions of C^*-algebras [Segal, Duke Math. J. **18**, 221–265 (1951); these Rev. **13**, 534]. The extensions replace C^*-algebras by arbitrary normed *-algebras with approximate identities in theorems about representations induced by traces and decompositions of stationary traces.

I. E. Segal (Chicago, Ill.).

Allen, H. S. Idempotent operators on a vector space. Quart. J. Math., Oxford Ser. (2) **3**, 94–97 (1952).

Theorems on algebras of operators $A, B, \cdots$ on a vector space α into itself. If $AB=I$, then $(I-BA)\alpha$ is the set of all $x\alpha$ with $Ax=0$. All left inverses L of B are given by $L=AC$, where C is an operator which leaves $\beta=B\alpha$ invariant and has an inverse. The general form of an idempotent (projector) with range β is BL. There are applications to systems of linear equations with infinitely many variables.

G. G. Lorentz (Toronto, Ont.).

Amemiya, Ichiro. On a topological method in semi-ordered linear spaces. Proc. Japan Acad. **27**, 138–140 (1951).

An abstract space R can be topologized in the following way: Let $\mathfrak{L}$ be an arbitrary fixed family of subsets (to be called cells) and call a subset W open if and only if, for every cell C, WC is the union of some collection, not necessarily countable, of cells. Assume also that πC_n is non-empty whenever $C_1 \supset C_2 \supset \cdots$ with non-empty cells C_n. The theorem then holds: if R is covered by a sequence of closed sets, $R=\sum B_n$, then at least one B_n contains a cell. For example, let R be a continuous, semi-ordered linear space, in other words, a conditionally complete, partially ordered vector space; let A be an arbitrary family of elements $a \geqq 0$ and let $\mathfrak{L}$ consist of all subsets $u \leqq x \leqq v$ with $v-u$ in A. If A has the property: a, b in A imply c is in A for some $c \leqq a \wedge b$, then a linear functional $L(x)$ on R is continuous with respect to the A-cell topology if and only if for every $\epsilon>0$, $|L(x)|<\epsilon$ on some cell $0 \leqq x \leqq a$. The theorem then follows: suppose $L_n(x)$ are all linear and converge for each x to a limit $L(x)$. If the $L_n(x)$ are continuous with respect to one or more given A-topologies on R then so is $L(x)$. By suitable choice of the family of A-topologies it follows that if the linear $L_n(x)$ are order-bounded, order-continuous, or order-universally-continuous respectively, then so is the limit.

I. Halperin (Kingston, Ont.).

Saito, Toshiya. On the measure-preserving flow on the torus. J. Math. Soc. Japan **3**, 279–284 (1951).

This paper aims to prove that the flow defined on the torus by the equations $dx/dt=X(x, y)$, $dy/dt=Y(x, y)$, where X and Y are of class C' and period 1 in x and y and satisfy $\partial X/\partial x+\partial Y/\partial y=0$, is ergodic with respect to plane measure if and only if (1) X and Y have no common zeros, and (2) the integrals of X and Y over the unit square are both different from zero and their ratio is irrational. However, the proof purporting to show the necessity of condition (1) is inconclusive, and examples show that this part of the stated result is in fact false. (It is tacitly assumed that any common zero is isolated and that the first degree terms in the expansions of X and Y are not all zero.) It is shown that (2) is necessary and that (1) and (2) are sufficient (at least in case X and Y are C'). For applications, see the author's paper, Kōdai Math. Sem. Rep. **1951**, 21–25 (1951); these Rev. **13**, 256.

J. C. Oxtoby (Bryn Mawr, Pa.).

Theory of Probability

***Kneale, William. Probability and induction.** Oxford, At the Clarendon Press, 1949. viii+264 pp.

This is a philosophical book on probability and induction. Contents: I) Knowledge and belief; II) The traditional problem of induction; III) The theory of chances; IV) The probability of inductive science.

J. Łoś (Toruń).

Katz, Leo. The distribution of the number of isolates in a social group. Ann. Math. Statistics **23**, 271–276 (1952).

Let $P(i)$ be the probability that exactly i "people" in a set of N remain "unchosen", when each "person" in the set "chooses" exactly d other "people" in it at random. A history of the study of $P(i)$ is given, exact formulae are derived for $P(i)$ and for the moments, and approximations are studied.

L. J. Savage.

Hernandez, Enrique Juan. Generalization of the probability laws of Laplace-Gauss and Cauchy. Trabajos Estadística **2**, 291–310 (1951); **3**, 139–167 (1952). (Spanish. French summary)

Santaló, Luis A. Probability in geometrical constructions. An. Soc. Ci. Argentina **152**, 203–229 (1951). (Spanish)

Numerous geometrical probabilities are evaluated. A typical result is: if numbers a, A, p are chosen independently and at random over the intervals $(0, K)$, $(0, \pi)$, $(0, K)$, the probability that there exists a triangle having one side a, the opposite angle A, and radius of inscribed circle p is $(2\pi)^{-1}\log 2=.110\cdots$. If a, A are chosen independently and at random over $(0, K)$, $(0, \pi)$ and $p=K-a$, the probability of the same event is

$$\frac{8}{5\pi}\left(\frac{\pi}{8}+\frac{3}{2}\log 2-\log 3\right)=.170\cdots.$$

D. Blackwell (Washington, D. C.).

Sukhatme, B. V. On certain probability distributions arising from points on a line. J. Roy. Statist. Soc. Ser. B. **13**, 219–232 (1951).

The paper discusses for m points of k different colors on a line the distributions of the number of (a) joins between points of different colors, (b) S-plets formed from points of one color, (c) runs of length r of points of the same color and the joint distribution of runs of length r_1 and r_2 of points of the same color. The distributions are obtained for free sampling and also for the case that the number of points of each color is fixed. The author obtains the first four cumulants of these distributions. The distributions tend to the normal distribution when m increases except when the probability for the points of a specified color is small, in which case they tend to the Poisson distribution.

H. B. Mann (Columbus, Ohio).

Tricomi, Francesco G. Applicazione della funzione gamma incompleta allo studio della somma di vettori casuali. Boll. Un. Mat. Ital. (3) **6**, 189–194 (1951).

Man betrachtet n Vektoren im euklidischen Raume von p ($\geqq 2$) Dimensionen, mit den Beträgen $a_1, a_2, \cdots, a_n$, Richtung und Sinn ganz beliebig. Die Verteilungsfunktion

$P_n(r_0)$ bedeutet die Wahrscheinlichkeit, dass ihre Summe $r \leqq r_0$ ist. Bekanntlich hat Kluyver gefunden:

$$P_n(r) = \frac{[\Gamma(\nu+1)]^{n-1} 2^{(n-1)\nu+1}}{(a_1 a_2 \cdots a_n)^\nu} \int_0^\infty t^{-(n-1)\nu} J_{\nu+1}(rt) \prod_{m=1}^{n} J_\nu(a_m t)\,dt$$

[vgl. G. N. Watson, Bessel functions, 2nd ed., Cambridge, 1944, p. 421; diese Rev. 6, 64]. Der Verfasser betrachtet den Fall $a_1 = a_2 = \cdots = a_n = 1$, wenn n gross ist, und findet:

$$\Gamma(p/2) P_n(r) = \int_0^x e^{-t} t^{p/2-1}\,dt - \frac{1}{n} e^{-x} x^{p/2} \left(\frac{1}{2} - \frac{x}{p+2} \right) + O(n^{-2})$$

wo $x = pr^2/2n$. Dieser Ausdruck erlaubt eine unmittelbare Bestimmung des Mittelwertes

$$\bar{r} = \int_0^\infty r \frac{dP_n}{dr}\,dr = \int_0^\infty [1 - P_n(r)]\,dr,$$

nämlich,

$$\bar{r} = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} \sqrt{\frac{2n}{p}} \left[1 + \frac{1}{4(p+2)n} + O(n^{-2}) \right].$$

S. C. van Veen (Delft).

Darling, D. A. The influence of the maximum term in the addition of independent random variables. Trans. Amer. Math. Soc. **73**, 95–107 (1952).

Soient $\{X_i\}$ $(i = 1, 2, \cdots)$ des variables aléatoires mutuellement indépendantes de même fonction de répartition $F(x)$ absolument continue avec $F(0) = 0$ (ces restrictions sur $F(x)$ peuvent être élargies pour certains des résultats suivants); $S_n = \sum_{i=1}^n X_i$, X_n le plus grand des nombres $X_1, X_2, \cdots, X_n$. a) Si pour tout $\alpha > 0$ on a

$$\lim_{x \to +\infty} (1 - F(\alpha x))/(1 - F(x)) = 1,$$

S_n/X_n^* tends vers 1 en probabilité et

$$\lim_{n \to +\infty} \Pr\{n - nF(S_n) < y\} = 1 - e^{-y};$$

b) s'il existe des nombres certains (a_n, b_n) tels que $a_n S_n + b_n$ ait une distribution limite stable d'exposant γ, on a: (1) si $0 < \gamma < 1$, $\lim_{n \to +\infty} \Pr\{S_n < y X_n^*\} = G(y)$, où $G(y)$ est une fonction de répartition dont l'auteur détermine la fonction caractéristique; (2) si $1 < \gamma < 2$, en posant $\mu = E(X_i)$, $1 - F(c_n) = 1/n$, on a

$$\lim_{n \to +\infty} \Pr\{S_n/X_n^* < ny/c_n\} = 1 - \exp -(x/\mu)^\gamma$$

et $\lim_{n \to +\infty} \Pr\{S_n - n\mu < X_n^* y\} = H(y)$, où $H(y)$ est une fonction de répartition dont l'auteur détermine la fonction caractéristique (l'auteur indique sans détails qu'on peut traiter aussi le cas $\gamma = 1$). *R. Fortet* (Caen).

Linnik, Yu. V. Linear statistics and the normal distribution law. Doklady Akad. Nauk SSSR (N.S.) **83**, 353–355 (1952). (Russian)

Let $x_1, \cdots, x_r$ be independent random variables having the same distribution $F(x)$; $L_1(x) = a_1 x_1 + \cdots + a_r x_r$, $L_2(x) = b_1 x_1 + \cdots + b_r x_r$ where the a and b are constants. A set of necessary and sufficient conditions is stated for the equivalence of the following two assertions: (A) $F(x)$ is a normal law; (B) $L_1(x)$ and $L_2(x)$ have the same distribution. The set of conditions are too long to give here; they depend on the nature of the roots of the entire function $|a_1|^s + \cdots + |a_r|^s - |b_1|^s - \cdots - |b_r|^s$. Marcinkiewicz [Math. Z. **44**, 612–618 (1938)] proved that if the a's are not a permutation of the $\pm b$'s, then (B) implies (A) provided that (C) all moments of $F(x)$ are finite; he raised the question if (C) could be weakened. It follows from the present theorem that the answer is no. The theorem can be used to reduce the test of normality to that of homogeneity (=having the same distribution). Numerical examples are given. No proofs are given. *K. L. Chung.*

Gnedenko, B. V., and Mihalevič, V. S. Two theorems on the behavior of empirical distribution functions. Doklady Akad. Nauk SSSR (N.S.) **85**, 25–27 (1952). (Russian)

Let $x_1, \cdots, x_n$; $y_1, \cdots, y_m$ be independent random variables having the distribution $F(x)$. Let $S_n(x)$ and $T_m(x)$ be the empiric distribution functions based on the x and y respectively. A point x_k at which $S_n(x_k - 0) \geqq T_m(x_k)$ is called a positive jump (of $S_n(x)$ with respect to $T_m(x)$). Let $C(n, m)$ be the number of positive jumps. Theorem 1. If $n = mp$ where p is a positive integer, then $P(C(n, m) = k) = n^{-1} + 1$ for $k = 0, 1, \cdots, n$. For $p = 1$ this was proved by the authors [same Doklady **82**, 841–843 (1952); these Rev. **13**, 760] and is equivalent to a result of Chung and Feller [see the cited review]. The present proof is based on a geometrical one-one correspondence and does not use the case $p = 1$. Theorem 2. The random variable Δ_m equal to the measure of the projection on the y-axis of the set of points $(x, F(x))$ for which $F(x) \geqq T_m(x)$ is uniformly distributed in $(0, 1)$ for every fixed m. Theorem 2 is proved by using Theorem 1 and the theorem of Glivenko-Cantelli. Kac [Proc. Nat. Acad. Sci. U. S. A. **35**, 252–257 (1949); these Rev. **10**, 614] had proved the assertion in Theorem 2 as $m \to \infty$. *K. L. Chung.*

Rényi, Alfréd. On a new generalization of the central limit theorem of probability theory. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei **1**, 351–355 (1951). (Hungarian)

Suppose that $\{\xi_n\}$ is a sequence of independent functions on the unit interval, such that each ξ_n takes on only a finite number of values and such that $\xi_n(x) = \xi_n(y)$ for all n implies that $x = y$. Suppose, moreover, that the mean of each ξ_n is 0 and that the variance B_n of $\sum_{k=1}^n \xi_k$ tends to infinity with n. If $A_n(x)$ is the probability that $\xi_1 + \cdots + \xi_n < B_n x$, and if μ' is a measure that is absolutely continuous with respect to Lebesgue measure μ, then

$$\lim_{n \to \infty} \mu'(A_n(x)) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{x} e^{-t^2/2}\,dt = \Phi(x).$$

Although the present paper contains no proofs, the author mentions that his proof rests on the following lemma: $\lim_{n \to \infty} \mu(A \cap A_n(x)) = \mu(A)\Phi(x)$ for every measurable set A. In a note added in proof the author says that, in a lecture at the first Hungarian Mathematical Congress, Kolmogoroff gave a far-reaching generalization of the author's result.

P. R. Halmos (Chicago, Ill.).

Hinčin, A. Ya. Limit theorems for sums of positive random quantities. Ukrain. Mat. Žurnal **2**, no. 4, 3–17 (1950). (Russian)

The author separates out of their physical context various probability theorems which he has applied to statistical mechanics [Mathematical foundations of statistical mechanics, Moscow-Leningrad, 1943, or, in translation, Dover, New York, 1949; these Rev. **8**, 187; **10**, 666]. For example: let Ω_n be the nth convolution with itself of the probability density of a non-negative random variable x with expecta-

tion a. It is supposed that x can take arbitrarily small values, and various regularity conditions are imposed on its density function. Suppose that $\alpha<a$ and that $\phi_n(\beta)=E\{e^{-\beta x}\}e^{n\beta}$. If $b'=d^2\log\phi_0(\beta)/d\beta^2$, and if β satisfies the equation $\alpha+\varphi'(\beta)/\varphi(\beta)=0$, it is shown that

$$\Omega_n(n\alpha+y)=\phi_n(\beta)^n e^{\beta y}\{(2\pi nb')^{-1/2}+O((1+y^2)/n^{3/2})\}.$$

The application to statistical mechanics is sketched. The result is extended to the iterated convolutions of a function whose integral may be infinite. For a special case of the result see an earlier paper of the author [Doklady Akad. Nauk SSSR 71, 1037–1039 (1950); these Rev. 11, 605].

J. L. Doob (Urbana, Ill.).

Gnedenko, B. V. On a local theorem for stable limit distributions. Ukrain. Mat. Žurnal 1, no. 4, 3–15 (1949). (Russian)

Proof of a theorem announced elsewhere [Doklady Akad. Nauk SSSR 66, 325–326 (1949); these Rev. 10, 720], giving a local theorem for the region of attraction of stable laws. The announced theorem covered only normal attraction.

J. L. Doob (Urbana, Ill.).

Meĭzler, D. G., Parasyuk, O. S., and Rvačeva, E. L. On a many dimensional local limit theorem of the theory of probability. Ukrain. Mat. Žurnal 1, no. 1, 9–20 (1949). (Russian)

Proof of results announced in another paper [Doklady Akad. Nauk SSSR 60, 1127–1128 (1948); these Rev. 10, 132].

J. L. Doob (Urbana, Ill.).

Lévy, Paul. Convergence des séries aléatoires et loi normale. C. R. Acad. Sci. Paris 234, 2422–2424 (1952).

Let X_k, $k=1, 2, \cdots$, be independent random variables with the common distribution $F(x)$. Let $H(x)=\int_0^x y^2dF(y)$; $S_n=\sum_{k=1}^n a_kX_k$. Suppose that $H(x)$ increases "regularly" to infinity and the a_k are positive and decrease "regularly". (The kind of regularity required is specified.) Moreover suppose that $\int_x^\infty y^{-2}dH(y)=o(x^{-2}H(x))$ as $x\to\infty$ so that $F(x)$ belongs to the domain of attraction of the normal law. Then the almost sure divergence of $\sum a_kX_k$ is a necessary and sufficient condition that S_n be asymptotically normal. These two circumstances may co-exist when $\sum a_n^2<\infty$. To every $H(x)$ there corresponds a cut in the sequences $\{\lambda(n)\}$ where $\lambda(n)=(na_n^2)^{-1}$. S_n is asymptotically normal or not according as $\{\lambda(n)\}$ belongs to the lower or upper class. The cut varies between those $\{\lambda(n)\}$ for which $\sum(n\lambda(n))^{-1}=\infty$ and the sequences $\{n^\beta\}$ $(\beta>0)$. As $H(x)$ increases more and more rapidly every increasing sequence $\{\lambda(n)\}$ such that $\log\lambda(n)=o(\log n)$ ultimately enters the lower class. Proofs are not given.

K. L. Chung (Ithaca, N. Y.).

Chung, Kai Lai, and Pollard, Harry. An extension of renewal theory. Proc. Amer. Math. Soc. 3, 303–309 (1952).

If $x_1, x_2, \cdots$ are independent identically distributed non-lattice random variables with

$$0<E(x_i)=m<\infty,\quad S_n=x_1+\cdots+x_n,\quad \text{and}\quad \limsup_{t\to\infty}|f(t)|<1,$$

where $f(t)$ is the characteristic function of the distribution of x_i, then

$$\sum_{k=1}^{\infty}P(x<S_n<x+h)\to\frac{h}{m},\ 0 \text{ as } x\to+\infty,\ -\infty.$$

An analogous result is announced for lattice variables. Purely analytic methods are used. [Cf. Chung and Wolfowitz, Ann. of Math. 55, 1–6 (1952); these Rev. 13, 475.]

D. Blackwell (Washington, D. C.).

Gihman, I. I. On some differential equations with random functions. Ukrain. Mat. Žurnal 2, no. 3, 45–69 (1950). (Russian)

The author studies the solutions of differential equations of the form $(x-\alpha)'=A(t, x)$. Here α is a given continuous function of t, a sample function of a stochastic process, and the solution will then define a second stochastic process. The sample functions of the α process will not in general be differentiable. The dependence of the solution on small changes in the α process and in the function A are studied in detail, and applied to the equations $x'=A(t/\epsilon)+f(\epsilon, t)$ and $x'=A(t, x)+f(t/\epsilon)$ (where f is a random variable for each pair of values of the indicated arguments). Under hypotheses on A and f too complicated to reproduce here, the qualitative properties of the $x(t)$ process in the limit when $\epsilon\to 0$ are investigated.

J. L. Doob (Urbana, Ill.).

Gihman, I. I. On the theory of differential equations of random processes. Ukrain. Mat. Žurnal 2, no. 4, 37–63 (1950). (Russian)

This is the first of two papers devoted to proving the results announced in an earlier paper [Doklady Akad. Nauk SSSR 58, 961–964 (1947); these Rev. 9, 293]. What is involved is a deep generalization of the (symbolic) differential equation

$$dx(t)=A[t, x(t)]dt+B[t, x(t)]da(t),$$

where A, B are functions of the indicated variables, and the $a(t)$ stochastic process has in the past been supposed to have independent increments. In the author's treatment this hypothesis is merely a special case which leads to solutions which are Markov processes. Moreover, in the author's treatment the random variables involved are unusually general, taking on values in a unitary space, and the above differential equation is replaced by a more abstract functional equation.

J. L. Doob (Urbana, Ill.).

Grenander, Ulf, and Rosenblatt, Murray. On spectral analysis of stationary time series. Proc. Nat. Acad. Sci. U. S. A. 38, 519–521 (1952).

Let $x_n=\sum_{j=-\infty}^{\infty}a_{n-j}\xi_j$, where the ξ_j's are mutually independent real random variables with a common distribution having zero mean, and the a_j's are real constants with $\sum_j a_j^2<\infty$. The x_n stochastic process is then stationary. The following results are announced, under further unspecified hypotheses on the a_j's and on the ξ_j distribution. Let F be the spectral distribution function of the x_n process, and let F_N^* be the integrated periodogram function of a sample of $x_1, \cdots, x_N$. Then if the ξ_j distribution has 3 as ratio of fourth moment to the square of the second, and if c is a certain constant depending on F, the asymptotic distribution of $c\max_{0\le\lambda\le\pi}|N^{1/2}[F_N^*(\lambda)-F(\lambda)]|$ is evaluated, $N\to\infty$. The limit distribution does not depend on F except by way of c, and is specified in terms of a known distribution involving the Brownian motion (Wiener) process. A function of $x_1, \cdots, x_N$ is given which converges stochastically to c, and which can be used to replace c in the above for statistical testing. A corresponding result holds for the difference between two independently obtained integrated periodo-

gram functions. If the above moment relation is not satisfied, the limit distribution still exists, but depends on the detailed structure of the spectral distribution function in a way which makes the result less useful for statistical applications.
J. L. Doob (Urbana, Ill.).

Radek, H. Ein Problem verketteter Wahrscheinlichkeiten. Österreich. Ing.-Arch. 6, 208–222 (1952).

Consider the elements $a_1, \cdots, a_s$ and $b_1, \cdots, b_n$. Each a picks exactly one b, the probability (pr) of each b being picked by each a being $1/n$ (this assumption is not explicitly stated), and kills it with a known constant pr w. The actions of the different a's are independent so that if r of the a's pounce on the same b, the pr. of the latter being killed is $1-(1-w)^r$. (This seems to be what the author means by "each b can be killed only once".) The pr $V(s, n, t, w)$ that exactly t of the b's are killed by the joint action of all the a's is shown to be $\binom{n}{t}\sum_{k=0}^{t}\binom{t}{k}(-1)^{t+k}[1-(n-k)w/n]^s$. An extension where each b also picks one a and kills it with pr w_2 is considered, and this mutual massacre is even iterated. It is assumed that the actions of the a's and the actions of the b's are independent (as in simultaneous shooting). Continuous analogue and numerical tables. *K. L. Chung.*

Takács, Lajos. Probabilistic treatment of the simultaneous stoppage of machines with consideration of the waiting time. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 1, 228–234 (1951). (Hungarian)

The following situation, considered earlier by Hinčin [Mat. Sbornik **40**, 119–123 (1933)], is discussed. A certain number of machines are supervised by one attendant. Each machine stops from time to time and needs repair. The time at which the stoppage occurs as well as the duration of repair are random variables. The author considers the period of operation $(0, t)$ and determines the expectations of (1) the time spent on repairs, (2) the number of stoppages, (3) the duration of stoppages. He states that his treatment differs from Hinčin's by (a) omitting the assumption of stationarity, (b) specifying the distribution of the duration of repairs. *E. Lukacs* (Washington, D. C.).

Hodge, P. G., Jr. The Brownian motion of coupled systems. J. Math. Physics **31**, 120–130 (1952).

The author studies the motion of an instrument in a fluid due to the impact of the molecules of the fluid. The mean square displacement is calculated for the case of stable equilibrium and for the case of neutral equilibrium. In the latter case the result is a linear function of time under certain restrictions on the length of the time intervals considered.
J. L. Snell (Princeton, N. J.).

Bhabha, H. J., and Ramakrishnan, A. The mean square deviation of the number of electrons and quanta in the cascade theory. Proc. Indian Acad. Sci., Sect. A. **32**, 141–153 (1950).

Let $R(E, E')dE'$ denote the probability per unit length that a particle (electron or positron) of energy E emits a photon of energy in the interval between E' and $E'+dE'$, and $R'(E, E')dE'$, the corresponding probability of a photon of energy E creating a pair whose electron (or positron) has an energy between E' and $E'+dE'$. Further let $T^{11}(E)$, $T^{12}(E)$, etc., stand for operators which acting on any function $f(E)$ of E turn it into

$$T^{11}(E)f(E) = \int_E^\infty R(E', E'-E)f(E')dE' - \int_0^\infty R(E, E')f(E)dE',$$

$$T^{12}(E)f(E) = 2\int_E^\infty R'(E', E)f(E')dE',$$

$$T^{21}(E)f(E) = 2\int_E^\infty R(E', E)f(E')dE',$$

$$T^{22}(E)f(E) = -\int_0^E R'(E, E')f(E)dE'.$$

Then, if $n(E; t)$ and $m(E; t)$ give, respectively, the mean number of particles and photons at a depth t in the material, the equations governing them are

$$\text{(1a)}\qquad \frac{d}{dt}n(E; t) = T^{11}(E)n(E, t) + T^{12}(E)m(E; t),$$

$$\text{(1b)}\qquad \frac{d}{dt}m(E; t) = T^{21}(E)n(E; t) + T^{22}(E)m(E; t).$$

The mean number of particles $\overline{N(E_1, E_2)}$ whose energies lie between E_1 and E_2 is then

$$\overline{N(E_1E_2)} = \int_{E_1}^{E_2} n(E; t)dE,$$

while the mean square number of particles $\overline{N(E_1, E_2)^2}$ is according to Bhabha's stochastic theory of continuous parametric systems [Proc. Roy. Soc. London. Ser. A. **202**, 301–322 (1950); these Rev. **12**, 191] given by

$$\overline{N(E_1, E_2)^2} - \overline{N(E_1, E_2)} = \int_{E_1}^{E_1}\int_{E_1}^{E_1} n_2(E, E'; t)dEdE',$$

where $n_2(E, E'; t)$ and the two other functions $nm(E, E'; t)$ and $m_2(E, E'; t)$ are determined by the equations

$$\begin{aligned}\text{(2a)}\qquad \frac{d}{dt}n_2(E, E'; t) &= T^{11}(E)n_2(E, E'; t)\\ &\quad + T^{11}(E')n_2(E, E'; t)\\ &\quad + T^{12}(E)nm(E', E; t)\\ &\quad + T^{12}(E')nm(E, E'; t)\\ &\quad + 2R'(E+E', E)m(E+E'; t),\end{aligned}$$

$$\begin{aligned}\text{(2b)}\qquad \frac{d}{dt}nm(E, E'; t) &= T^{11}(E)nm(E, E'; t)\\ &\quad + T^{12}(E)m_2(E, E'; t)\\ &\quad + T^{21}(E')n_2(E, E'; t)\\ &\quad + T^{22}(E')nm(E, E'; t)\\ &\quad + R(2E-E', E)n(2E-E'; t),\end{aligned}$$

$$\begin{aligned}\text{(2c)}\qquad \frac{d}{dt}m_2(E, E'; t) &= T^{11}(E)nm(E, E'; t)\\ &\quad + T^{21}(E')nm(E', E; t)\\ &\quad + T^{22}(E)m_2(E, E'; t)\\ &\quad + T^{22}(E')m_2(E, E'; t).\end{aligned}$$

In the practical applications the functions $R(E, E-E')$ and $R'(E, E')$ have the forms

$$R(E, E') = \left\{\frac{E'}{E} - \left(\frac{4}{3}+\alpha\right)\left(1-\frac{E}{E'}\right)\right\}\frac{1}{E},$$

$$R'(E, E') = \left\{1-\left(\frac{4}{3}+\alpha\right)\left(\frac{E'}{E}-\frac{E'^2}{E^2}\right)\right\}\frac{1}{E},$$

where α is a constant. The authors show how in this case the symbols of equations (1) can be solved by applying the Mellin transformation

$$\nu(r; t) = \int_0^\infty E^{r-1}n(E, t)dE,$$

$$\gamma(r, t) = \int_0^\infty E^{r-1}m(E, t)dE,$$

while the system of equations (2) can be solved by applying a double Mellin transformation

$$\nu_2(r, s, t) = \int_0^\infty dx \int_0^\infty dy x^{r-1}y^{s-1}n_2(x, y; t), \text{ etc.}$$

For, after applying the transformations, one obtains systems of linear equations with constant coefficients which can be solved by elementary methods. Finally, by applying the inverse Mellin transformation to the solutions obtained in this manner, we can get the required solutions of the physical problem. *S. Chandrasekhar.*

Bhabha, H. J. Note on the complete stochastic treatment of electron cascades. Proc. Indian Acad. Sci., Sect. A. **32**, 154–161 (1950).

In this paper the author shows how in terms of the two independent sets of solutions of equations (1) of the preceding review which corresponds respectively to showers started by an electron and a photon of some given energy E_0 [i.e., solutions which satisfy the boundary conditions $n(E_0, E; 0) = \delta(E-E_0)$, $m(E_0, E; 0) = 0$ and $n(E_0, E; 0) = 0$, $m(E_0, E; 0) = \delta(E-E_0)$ where δ denotes Dirac's δ-function] one can obtain the complete solution of the underlying stochastic problem. The details are somewhat too involved to be described here. *S. Chandrasekhar.*

Mathematical Statistics

Pompilj, G. Osservazioni sull'omogamia: La trasformazione di Yule e il limite della trasformazione ricorrente di Gini. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **9**, 367–388 (1950).

The author discusses certain transformations of two-by-two contingency tables and shows that any measure of association which is invariant under these transformations must be a function of $r_{11}r_{22}/r_{12}r_{21}$ where r_{ij} $(i, j = 1, 2)$ is the frequency in the cell (i, j). *E. Lukacs.*

Hartley, H. O., and Fitch, E. R. A chart for the incomplete beta-function and the cumulative binomial distribution. Biometrika **38**, 423–426 (1951).

Six nomograms (all on one page) and a calibrated ruler make it possible to solve the equation $P = I_x(a, b)$ for P when x is given, or for x when P is given. The charts are prepared for the following values of the parameters a, b: $2 \leqq a \leqq 200$ and $1 \leqq b \leqq 4$; $5 \leqq a \leqq 200$ and $4 \leqq b \leqq 15$; $20 \leqq a \leqq 200$ and $15 \leqq b \leqq 60$. The theory is based on an approximate formula due to Wise [Biometrika **37**, 208–218 (1950); these Rev. **12**, 724]. No estimates of accuracy are given, but numerical examples indicate that two significant digits can be read off correctly. *Z. W. Birnbaum.*

Uranisi, Hisao. The distribution of statistics drawn from the Gram-Charlier type A population. Bull. Math. Statist. **4**, 1–14 (1950).

Let $f(x)$ be represented by a Gram-Charlier Type A series of the form

$$(1) \qquad f(x) = \frac{e^{-\frac{1}{2}x^2}}{(2\pi)^{1/2}}\left\{1+\sum_{i=1}^{\infty} a_i H_i(x)\right\}.$$

The author derives the distribution of Student's

$$t = (\bar{X}-m)(n-1)^{1/2}/s,$$

and essentially $s^2 = \sum_{i=1}^n (X_i-\bar{X})^2/n$ in random samples of n from (1) where $\bar{X} = \sum X_i/n$. Similar results are obtained for the variance ratio (F test) for two populations of type (1). In the case of t, numerical results are given for samples of 5, 10, 15, 21 both for one-sided and two-sided tests. Apparently the author is not familiar with the related results of Gayen [Biometrika **36**, 353–369 (1949); **37**, 236–255 (1950); these Rev. **11**, 447; **12**, 345]. Gayen used an Edgeworth series which conforms more closely to the asymptotic theory of distributions, and provided more extensive numerical tables for both t and F. *L. A. Aroian.*

Massey, Frank J., Jr. Distribution table for the deviation between two sample cumulatives. Ann. Math. Statistics **23**, 435–441 (1952).

Benderskiĭ, A. M. On the distribution of the absolute value of the maximum deviation from the mean in a series of observations. Doklady Akad. Nauk SSSR (N.S.) **85**, 5–8 (1952). (Russian)

Let $z_1, \cdots, z_n$ be n independent random variables with a common normal distribution, and let

$$\bar{z} = n^{-1}\sum_1^n z_i, \quad S^2 = n^{-1}\sum_1^n (z_i-\bar{z})^2,$$

$$v_n = \max_{1\leq i\leq n} |z_i-\bar{z}|S^{-1}, \quad F(\lambda) = \Pr\{v_n \geqq \lambda\}.$$

It is shown that

$$(*) \quad F(\lambda) = \frac{2n}{[\pi(n-1)]^{1/2}}\frac{\Gamma((n-1)/2)}{\Gamma((n-2)/2)} \times \int_\lambda^{(n-1)^{1/2}} \left(1-\frac{\tau^2}{n-1}\right)^{(n-4)/2} d\tau$$

if $(n/2)^{1/2} < \lambda \leqq (n-1)^{1/2}$. Explicit expressions for $F(\lambda)$ are given for $\lambda > [n(n-3)/(3n-8)]^{1/2}$, and it is found that the right-hand side of (*) provides a close approximation for $F(\lambda)$ if $n \leqq 25$ and $F(\lambda) \leqq 0.1$. Percentage points for the corresponding test can be obtained from those for $\max (z_i-\bar{z})S^{-1}$, which were tabulated by N. Smirnoff [C. R. (Doklady) Acad. Sci. URSS **33**, 346–350 (1941); these Rev. **5**, 127] and F. E. Grubbs [Ann. Math. Statistics **21**, 27–58 (1950); these Rev. **11**, 527], where the significance levels have to be doubled. *W. Hoeffding* (Chapel Hill, N. C.).

Sarmanov, O. V. On functional moments of a symmetric correlation. Doklady Akad. Nauk SSSR (N.S.) **84**, 887–890 (1952). (Russian)

Sarmanov, O. V. On functional moments of a nonsymmetric correlation. Doklady Akad. Nauk SSSR (N.S.) **84**, 1139–1142 (1952). (Russian)

This is an extension of previous results of the author on the order of magnitude of a line of regression [same Doklady **59**, 1061–1064; **60**, 545–548 (1948); these Rev. **9**, 442, 593]. Let $F(x, y)$ be a bivariate probability density,

$$p(x)=\int_{-\infty}^{\infty}F(x, y)dy, \quad P(y)=\int_{-\infty}^{\infty}F(x, y)dx,$$

and suppose that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{F^2(x, y)}{p(x)P(y)}dxdy<\infty.$$

Let $r(x)$ be an unbounded function (that is, for every $M>0$ the inequality $|r(x)|>M$ holds on a set of positive measure) such that

$$\int_{-\infty}^{\infty}r^2(y)\frac{F(x, y)}{p(x)}dy<\infty, \quad \int_{-\infty}^{\infty}r^2(x)\frac{F(x, y)}{P(y)}dx<\infty.$$

Let

$$\phi(x)=\int_{-\infty}^{\infty}r(y)\frac{F(x, y)}{p(x)}dy, \quad \psi(y)=\int_{-\infty}^{\infty}r(x)\frac{F(x, y)}{P(y)}dx.$$

In the first paper it is shown that if $F(x, y)=F(y, x)$, the function $\phi(x)=\psi(x)$ cannot satisfy the inequality $|\phi(x)|\geqq\lambda(|r(x)|-A)$ for all x such that $|r(x)|\geqq A$, where $\lambda>1, A>0$. In the second note it is proved that in the general case we cannot have simultaneously $|\phi(x)|\geqq\tau(|r(x)|)$ if $|r(x)|\geqq A$ and $|\psi(y)|\geqq\tau^{-1}[(1+\mu)|r(y)|]$ if $|r(y)|\geqq B$, where A, B, and μ are arbitrary positive numbers and $\tau(z)$ an arbitrary function such that for $z\geqq 0$: (I) $\tau(0)=0$; (II) $\tau(z)$ is monotonically increasing; (III) $\tau(z)$ is continuous (continuous on the right at $z=0$); (IV) either $\tau(z)$ or its inverse, $\tau^{-1}(z)$, is convex. *W. Hoeffding.*

Olds, Edwin G. A note on the convolution of uniform distributions. Ann. Math. Statistics **23**, 282–285 (1952).

Let $X_1, X_2, \cdots$ be a sequence of independent random variables such that X_i has rectangular distribution in $(0, a_i)$, with $a_i>0$, $i=1, 2, \cdots$. The author derives expressions for the probability density and the cumulative probability function of $s_n=X_1+\cdots+X_n$, and shows that the condition $\lim_{n\to\infty}\{\max_{k\leq n}a_k(\sum_{i=1}^{n}a_i^2)^{-1/2}\}=0$ is necessary and sufficient for the asymptotic normality of s_n. Applications to quality control are suggested. *Z. W. Birnbaum.*

Terpstra, T. J. The asymptotic normality and consistency of Kendall's test against trend, when ties are present in one ranking. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 327–333 (1952).

A ranking coefficient T is defined which is shown to be equivalent to Kendall's S when ties are present in one ranking. It is shown that, on the hypothesis that trend is absent, T is in the limit normally distributed and consistent. *P. Whittle* (Uppsala).

Dalenius, T., and Gurney, M. The problem of optimum stratification. II. Skand. Aktuarietidskr. **34**, 133–148 (1951).

The problem considered is the following: given the frequency function of a variate s in an infinite population, what is the best point of division of this frequency function into two strata for stratified random sampling. The variate y whose population mean is to be estimated from the sample is assumed to be related to s by the equation $y=\psi(s)+\eta$ where ψ and η are uncorrelated and $E(\eta)=0$. The division points are established for (i) proportional allocation, (ii) optimum allocation for fixed sample size, (iii) optimum allocation with cost C_i per unit in the ith stratum, (iv) ratio estimates. For (i) and (ii), division points are also given for finite populations, with illustrations which suggest that the results for an infinite population will nearly always be adequate in practice. *W. G. Cochran* (Baltimore, Md.).

Sillitto, G. P. Interrelations between certain linear systematic statistics of samples from any continuous population. Biometrika **38**, 377–382 (1951).

Linear systematic statistics are linear combinations of order statistics. The expectations of some members of a certain class of such statistics are obtained in terms of the expectations of others. Certain linear systematic statistics are pointed out which have expectations independent of the size of sample from which they are calculated, and some statistics of this type are derived for use as indices of skewness and kurtosis. *D. F. Votaw, Jr.*

Homma, Tsuruchiyo. On the asymptotic independence of order statistics. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. **1**, no. 1, 1–8 (1951).

Let $X_1<X_2<\cdots<X_n$ be an ordered sample of a random variable with a continuous cumulative distribution function. The purpose of this paper is to give a rigorous proof and a generalization of Gumbel's results on the asymptotic independence of X_r and X_s, with r, s fixed or dependent on n [Ann. Math. Statistics **17**, 78–81 (1946); these Rev. **7**, 464] and a necessary and sufficient condition for $a_nX_r+b_n$ to be asymptotically normal. *Z. W. Birnbaum.*

Ehrenberg, A. S. C. On sampling from a population of rankers. Biometrika **39**, 82–87 (1952).

A few remarks are made on a variety of ranking procedures, some of which have been previously treated in the literature and some of which are suggested for the first time by the author. A semi-heuristic and intuitive attempt is made to justify using certain coefficients of agreement among rankers in preference to others. *B. Epstein.*

Lloyd, E. H. On the estimation of variance and covariance. Proc. Roy. Soc. Edinburgh. Sect. A. **63**, 280–289 (1952).

Suppose we have a number of independent pairs of observations (X_i, Y_i) on two correlated variates (X, Y), which have constant variances and covariance, and whose expected values are of known linear form, with unknown coefficients, say $\sum_j p_{ij}a_j$, $\sum_j q_{ij}b_j$, respectively. The p_{ij} and the q_{ij} are known, the a_j and the b_j are unknown. The paper discusses the estimation of the coefficients, and of the variances and the covariance, and evaluates the sampling variances of the estimates. The argument is entirely free of distributional assumptions. (From the author's summary.) *B. Epstein* (Detroit, Mich.).

Guest, P. G. The estimation of standard error from successive finite differences. J. Roy. Statist. Soc. Ser. B. **13**, 233–237 (1951).

The author concerns himself with the estimation of the standard error in variate-difference calculations. The relative efficiencies of the estimates based upon mean differ-

ences and mean square differences are calculated, results which are subsequently applied to the estimation of standard error in polynomial fitting. *P. Whittle* (Uppsala).

Sandelius, Martin. Unbiased estimation based on inverse hypergeometric sampling. Kungl. Lantbrukshögskolans Annaler **18**, 123–127 (1951).

Drawings without replacement are made from a population of size N, consisting of N_a a's and $N-N_a$ b's. For estimating the parameter $p=N_a/N$, two procedures are considered: (i) sampling is continued until a prescribed number M (assumed to be $\leqq N_a$) of a's is obtained; (ii) sampling is continued until either M a's are obtained, or a prescribed sample size $n>M$ is reached. For each procedure an unbiased estimate of p and an unbiased estimate of this estimate are given. *Z. W. Birnbaum* (Seattle, Wash.).

Lloyd, E. H. Least-squares estimation of location and scale parameters using order statistics. Biometrika **39**, 88–95 (1952).

Consider distributions depending only on a location and a scale parameter. Let $X_i, i=1, 2, \cdots, n$, be a sample of n independent observations on a random variable X having such a distribution and let $Y_1 \leqq Y_2 \leqq \cdots \leqq Y_n$ be a rearrangement of the X_i's in ascending order. Linear combinations $\sum_{i=1}^{n} a_i Y_i$ which are unbiased and minimum variance are derived. Variances and covariances of the estimate are also obtained. If the underlying family of distributions is symmetric, it is shown that the estimates of location and scale respectively are uncorrelated. As special cases of symmetry the author treats the rectangular and normal distribution. The main result of this paper can be stated as follows: "The 'ordered' estimate of the population mean has a sampling variance which never exceeds that of the sample mean; it is equal to that of the sample mean if and only if the row-totals of the variance matrix of the ordered observations all have the same sum. The 'ordered' estimate then becomes the sample mean itself. When the variance matrix does not satisfy this condition the 'ordered' estimate of the population mean has strictly smaller sampling variance than has the sample mean." *B. Epstein* (Detroit, Mich.).

Halperin, Max. Maximum likelihood estimation in truncated samples. Ann. Math. Statistics **23**, 226–238 (1952).

The random variable X is assumed to have a probability density belonging to a one-parametric family $f(X;\theta)$. A sample of size n is obtained and, for given q, $0<q<1$, $r=[qn]$, only the r smallest values $X_1, \cdots, X_r$ of the sample are recorded. The author shows that when this information is used, the likelihood equation for θ has, under certain regularity assumptions on $f(X;\theta)$, a root $\hat{\theta}$ which converges in probability to θ, is asymptotically normally distributed, and asymptotically efficient. It is stated that this result can be generalized to the case of several parameters. Maximum likelihood estimates for finite n, r are given for special cases. *Z. W. Birnbaum* (Seattle, Wash.).

Barnard, G. A. The frequency justification of certain sequential tests. Biometrika **39**, 144–150 (1952).

Let $H_i, i=1, 2$, be the composite hypothesis that the chance variables $x_1, x_2, \cdots$ have the normal distribution with variance σ^2 and mean $\sigma\delta_i$. If one tests H_1 against H_2 sequentially using the procedure of Wald [Wald, Sequential analysis, Wiley, New York, 1947, pp. 81–84, especially p. 84; these Rev. **8**, 593] with uniform weight function in the range $a<\sigma<b$ for both H_1 and H_2, then, as was first pointed out by Arnold and Goldberg [see, for example, the introduction to "Tables to facilitate sequential t-tests", Nat. Bur. Standards Appl. Math. Ser., no. 7, U. S. Govt. Printing Office, Washington, D. C., 1951; these Rev. **13**, 141], the probability ratio criterion approaches, as $a\to 0$ and $b\to\infty$, the quotient λ_{n-2} of the density of a non-central t-distribution to that of a central t-distribution, both with $(n-2)$ degrees of freedom, where n is the number of observations. The present author uses the weight function $1/\sigma$ and, in the limit as $a\to 0$ and $b\to\infty$, obtains λ_{n-1}, the same quotient but with $(n-1)$ degrees of freedom (which seems to the reviewer much more reasonable). Since λ_{n-1} is invariant under the transformation $x\to kx$, it follows immediately, as the author points out, that the probabilities of error under either H_1 or H_2 when λ_{n-1} is the sequential test criterion do not depend on σ. (This is obviously also true of the distribution of the number of observations required to reach a decision.) The author then proves that the probabilities of error under both H_1 and H_2 (when λ_{n-1} is the sequential test criterion) are given approximately by Wald's formulae [ibid., pp. 41–42]. [The proof contains a gap in that the validity of the inversion of integration and passage to the limit, which occurs at the top of page 147, is not proved. A personal communication from the author promises that a proof of this will appear shortly. Also, in equation (7), s/b and s/a have been incorrectly written as sb and sa.] The author then shows that this same argument (and consequently the same conclusion) apply whenever there exists a suitable group of transformations which leave invariant the hypotheses to be tested. Finally, it is claimed that this result helps to reconcile several theories of statistical inference. Here the argument is philosophical and unintelligible to the reviewer. *J. Wolfowitz* (Los Angeles, Calif.).

Jordan, Károly. Statistical inference. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei **1**, 218–227 (1951). (Hungarian)

In the first part of the paper the author introduces a "surprise index" which he uses to test hypotheses. This procedure amounts to controlling the type 1 error by constructing a critical region which contains the points of the sample space with small probabilities. This approach is used to introduce the chi-square test and is somewhat similar to a procedure proposed by T. C. Fry [J. Amer. Statist. Assoc. **33**, 513–525 (1938)]. The last part of the paper contains a discussion of the gamma and beta distributions and their specializations. *E. Lukacs* (Washington, D. C.).

Špaček, A. Note on minimax solutions of statistical decision problems. Colloquium Math. **2** (1951), 275–281 (1952).

Apparently unaware that the result is well known and obvious, the author gives an elaborate proof of the fact that a zero-sum two-person game is determined if it is strictly determined (i.e., using pure strategies only). *J. Wolfowitz* (Los Angeles, Calif.).

Miyasawa, Kôichi. On the statistical decision function. I. Bull. Math. Statist. **4**, 22–32 (1950).

A necessary and sufficient condition that a sequential sampling plan should be Bayes is that, except for sets of measure zero, it must (a) continue when the optimum risk is less than the stopping risk, (b) stop when the optimum continuation risk exceeds the stopping risk, and (c) choose

a terminal decision yielding the stopping risk when sampling stops. *D. Blackwell* (Washington, D. C.).

Das, A. C. On the estimation of parameters in a recursive system. Sankhyā 11, 273–280 (1951).

An account of some sampling experiments performed upon certain evolutive economic models. Least square estimates of the model parameters appear to be satisfactory.

P. Whittle (Uppsala).

Sastry, A. Sree Rama. Bias in estimation of serial correlation coefficients. Sankhyā 11, 281–296 (1951).

Expressions are given for the expected value of the bias in the empirical autocorrelation coefficient caused by the correction for the mean. Numerical results are given for representative cases. *P. Whittle* (Uppsala).

Sastry, A. Sree Rama. Some moments of moment statistics and their use in tests of significance in autocorrelated series. Sankhyā 11, 297–308 (1951).

Let $x_1, x_2, \cdots, x_T, y_1, y_2, \cdots, y_T$ represent two stationary time series; some moments of their moment statistics are derived by straightforward but cumbersome algebra. Various results are given for the case where no assumptions are made about the distributions and where the processes are normal. Two modifications of the usual t-test for the hypothesis $E(x_i)=0$ are obtained in a heuristic fashion; the usual test [valid if the x_i are independent] and the two modifications are compared by empirical sampling from five experimental models. The modified statistics unfortunately involve population parameters. *D. G. Chapman.*

Kendall, M. G. Regression, structure and functional relationship. II. Biometrika 39, 96–108 (1952).

Comments on the rationale of (A) the "error-in-variable" and (B) the "error-in-equation" approaches. Stressing that the applications thus far reported do not suffice for an appraisal of (B) [see L. R. Klein, Economic fluctuations in the United States, 1921–1941, Wiley, New York, 1950; C. Christ, Conference on Business Cycles, Nat. Bur. Economic Res., New York, 1951, pp. 35–129], the author suggests for further investigation three groups of questions referring to the implications of the assumptions made in (B) concerning the properties of the disturbances. [The reviewer would add that there is an obscurity in (B) in the very approach. Since the relations are not ordinary regressions, and since disturbances are involved, a relation (B) cannot be optimal for the purpose of predicting or estimating one variable if the other ones are known. But then, supposing that unbiased estimates of the parameters have been obtained (by some or other of the methods devised for approach (B)), exactly what use can be made of the relation arrived at? In (A) this difficulty in the interpretation does not arise, since the hypothetical relation between the error-free variables is assumed to be exact.] *H. Wold.*

Bartlett, M. S. The effect of standardization on a χ^2 approximation in factor analysis. (With an appendix by W. Ledermann.) Biometrika 38, 337–344 (1951).

The author previously proposed a χ^2 analysis to test the significance of the successive roots of the determinantal equation $|R-\lambda I|=0$, where R is a matrix of sample correlations [British J. Psychology, Statist. Sect. 3, 77–85 (1950); 4, 1–2 (1951)]. The present paper considers the case with common variance and true correlations equal and not zero, and shows that the number of degrees of freedom left after a component has been eliminated depends on the true correlation. This complication does not arise in the unstandardized case with $|V-\lambda I|=0$, where V is the matrix of variances and covariances, each divided by either the true variance or the mean sample variance. All results are asymptotically valid for large samples. The appendix gives the characteristic equation and roots of a certain symmetric $(\frac{p}{2})$ by $(\frac{p}{2})$ matrix. *S. W. Nash* (Vancouver, B. C.).

***Finney, D. J. Probit analysis. A statistical treatment of the sigmoid response curve.** 2d ed. Cambridge, at the University Press, 1952. xiv+318 pp. $7.00

Probit analysis is used to estimate the relationship between the dose of some material, e.g., a drug, and the percentage of subjects affected by it, e.g., percent kill. In many cases, the percent kill is closely approximated by a cumulative normal distribution in terms of the logarithm of the dose. Suppose a given dose d results in a P percent kill, then the probit representation is (neglecting experimental error)

$$y=a+bx=\bar{y}+b(x-\bar{x}),$$

where $P/100=(2\pi)^{-1/2}\int_{-\infty}^{y-5}e^{-u^2/2}du$, $x=\log d$, and y is the probit. If two materials produce parallel responses (the b's are equal), their relative potencies can be compared by use of the two values of a. One of the important values to be estimated is the median effective dose (ED50), which can be obtained from $\log(\text{ED50})=\bar{x}+(5-\bar{y})/b$. A χ^2-test is used for goodness-of-fit. The probit transformation is used to linearize the response. Since the variance of y is not constant, a rather complicated weighted regression technique must be used.

Finney presents a detailed computing scheme in the appendix for determining a, b, log (ED50), standard errors, fiducial limits and a test of goodness-of-fit. Many other worked-out examples are included in the text. Methods are also presented to adjust for natural mortality during the course of the experiment. An excellent set of tables is available to make the computing easy. One chapter is devoted to graded response data, another to methods of comparing the relative potencies of several drugs and a third to problems connected with mixtures of poisons. The following topics are treated in a chapter on "Recent Developments": Poisson or negative binomial rather than binomial variation, incomplete replication in complex experiments, use of inverse and staircase sampling in determining size of sample and number of dosage levels to use, and a discussion of the need for replicated experiments to obtain better estimates of error.

This edition has the same basic methodology as the first edition, the only change being the addition of the chapter on "Recent Developments", more discussion of other methods and some minor corrections. Hence, reviews of that first edition are still pertinent, e.g., these Rev. 8, 592; J. Amer. Statist. Assoc. 43, 145–153 (1948).

The reviewer is seizing this opportunity to encourage a comparative study of the various methods of analyzing bioassay data, with special emphasis paid to these points: (1) Under what circumstance will the simpler variance stabilizing angle transformation produce different results from the various linearizing transformations, which require weighted analyses? Would better experimental designs be possible if a simpler analysis were used? (2) Why use 0 or 100 percent kill data? How can we use a transformation, such as the probit, to estimate the dosage needed for a very high kill, when this transformation is based on the assumption that a 100% kill can be produced only by an infinite dose? (3) Is it valid to assume that one has an infinite num-

ber of degrees of freedom for the estimates of error in probit analysis, even when χ^2 is non-significant? (4) Is it advisable to propagate the idea of rejecting observations (as in Table 12 of the present book) in order to obtain parallel response patterns? *R. L. Anderson* (Raleigh, N. C.).

Clarke, Pamela M. An analysis of rectangular lattices with unequal block sizes, using inter-block information. Biometrics 7, 287–294 (1951).

The author gives the analysis of rectangular lattices with unequal sides p, q recovering interblock information. The formulae for the variances of the differences between varietal means are approximations, but the author believes that the error is negligible as long as $|p-q| \leqq 5$ and $|p-q| \leqq pq/10$. Evidence from actual data supporting this belief is quoted in the paper. *H. B. Mann.*

Bose, R. C., and Shimamoto, T. Classification and analysis of partially balanced incomplete block designs with two associate classes. J. Amer. Statist. Assoc. 47, 151–184 (1952).

Partially balanced incomplete block designs were first defined by Bose and Nair [Sankhyā 4, 337–372 (1939)]. The definition in the present paper differs slightly from the original definition in that the frequencies λ_i with which pairs of different association classes occur are allowed to be equal. The paper is an exposition of remarkable clarity of the subject in the title. The authors distinguish 5 different classes of designs according to the association scheme. Illustrative examples are provided for each of these classes. The analysis is completely described using intrablock information only and using intrablock and interblock information combined. Several well chosen examples conclude the paper. *H. B. Mann* (Columbus, Ohio).

Banerjee, K. S. Weighing designs and partially balanced incomplete blocks. Calcutta Statist. Assoc. Bull. 4, no. 13, 36–38 (1951).

It is shown in this paper that in certain situations partially balanced incomplete block designs [Bose and Nair, Sankhyā 4, 337–372 (1939)] can be used for obtaining weighing designs. All the examples given by the author use only one subclass of p.b.i.b. designs, namely the group divisible [see the paper reviewed above]. *R. C. Bose* (Chapel Hill, N. C.).

Patterson, H. D. Change-over trials. J. Roy. Statist. Soc. Ser. B. 13, 256–270; discussion: 270–271 (1951).

The author discusses experimental designs in which treatments are applied to experimental units in succession without an intermediate rest period, so that to the treatments must be ascribed also a residual effect extending into successive periods. Such experiments are called change-over trials. In the designs discussed by the author the residual effect is assumed to extend only into the next experiment. Such designs may be arranged into complete sets of orthogonal Latin squares, where the columns are the experimental units arranged in blocks, the rows the periods of treatment, and the numbers the treatments. Further balances may be introduced by arranging the treatments in such a way that each treatment follows each other treatment an equal number of times. Besides the least square analysis the author also discusses an analysis based on minimizing a certain weighted sum of squares. If the number of treatments is large, Youden squares may be used instead of Latin squares; also some of the balanced and partially balanced incomplete block designs can be utilized. The designs are also applicable to factorial experiments. *H. B. Mann.*

TOPOLOGY

Tutte, W. T. The factors of graphs. Canadian J. Math. 4, 314–328 (1952).

This paper is concerned with the following question: Under what circumstances does a locally finite graph contain a subgraph which includes all the vertices of the graph and which has an assigned number of edges coming together at each vertex? If G is a graph and $f(x)$ is a function defined over the vertices x which takes positive integral values, then a subgraph of G embracing all the vertices of G, in which the degree of the vertex x is $f(x)$ for all x, is called an f-factor of G. In the special case where $f(x)=n$ for all x, the f-factor is a factor of degree n in the ordinary sense. The theory of factorization presented is therefore a natural generalization of the usual theory of factors. The main theorem obtained is a criterion for the existence of an f-factor.

If G is a locally finite graph and if S is a finite set of vertices of G, then $G(S)$ denotes the graph obtained by deleting from G all the vertices of S and all edges having one or both ends in S. If H is any finite connected component of $G(S)$, the number of edges which have one end in S and the other a vertex of H is $v(H)$. Let $K(G, S)$ be the set of all finite components H of $G(S)$ such that $v(H)+\sum_{c\in H} f(c)$ is an odd number. If $K(G, S)$ is finite, the number of its elements is denoted by $k(G, S)$. If S includes all the vertices of G, $k(G, S)$ is defined to be zero. The degree of the node c in G is $d_G(c)$. In this notation the "recalcitrance" of G with respect to S, $r(G, S)$, is defined by

$$r(G, S)=k(G, S)+\sum_{c\in S}(f(c)-d_G(c)).$$

If $K(G, S)$ is infinite, $r(G, S)$ is defined to be infinite also. Next, the graph G is said to be "constricted" with respect to the function f if there exist disjoint finite sets of vertices S and T, possibly null, such that

$$\sum_{c\in T} f(c) < r(G(T), S).$$

(In particular, G is constricted if it has a vertex a such that $d_G(a)<f(a)$ or if $\sum_{c\in G} f(c)$ is odd.)

The paper is mostly devoted to proving that G has no f-factor if and only if G is constricted with respect to f. In the case $f(x)=1$ for all x, this condition is not so simple as the one given by the author in two earlier papers, but it is shown to reduce to this simpler condition by means of a result analogous to the main theorem. This result is a criterion for the graph to be f-soluble. It is f-soluble if it is possible to associate with each edge A a non-negative integer $h(A)$ so that for each vertex x the sum of the numbers $h(A)$, taken over all edges A of G having x as an end, is $f(x)$. *G. A. Dirac* (Toronto, Ont.).

Cantoni, Riccardo. Conseguenze dell'ipotesi del circuito totale pari per le reti con vertici tripli. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 14(83), 371–387 (1950).

It is well known that a cubic planar graph which can be factored into three first-degree factors forms the set of boundaries of a map colorable in four colors. This paper considers other consequences of the hypothesis of factorability

for graphs (which need not be planar). Let F_1, F_2, F_3 be the three factors of the cubic graph G, possessing $2n$ vertices. It is always possible to mark just n of the vertices of G, so that every 1-cell of F_2+F_3 has one end on a marked vertex and the other end on an unmarked vertex. Such 1-cells are called pure, while the impure 1-cells are those whose ends are either both marked or both unmarked. The impure cells can thus occur only in F_1, although in general F_1 also contains pure cells. Next we consider a second marking in which all the cells of F_1+F_3 are pure. A fundamental theorem of the paper asserts that the set of cells which are pure with respect to both markings fall into p isolated sets (where the positive integer p can be 1 only in case all cells are pure with respect to both markings) such that the cells which are impure with respect to either marking never join two vertices of the same set but rather always join vertices belonging to isolated sets. These cells are thus said to form a "cincture" of G. From the same factorization of G we can in general obtain two other cinctures of G by two other pairs of sets of markings. By extending ideas of this nature the author obtains a classification of all factored cubic graphs. The same graph can belong to different classifications if it can be factored in more than one way. *D. C. Lewis.*

Cantoni, R. Una rete speciale. Period. Mat. (4) **29**, 37–41 (1951).

An example of a non-planar cubic graph without leaves which cannot be factored into three first-degree factors [cf. the following review.]. *D. C. Lewis* (Baltimore, Md.).

Bäbler, F. Bemerkungen zu einer Arbeit von Herrn R. Cantoni. Comment. Math. Helv. **26**, 117–118 (1952).

A simplified production of an example showing the existence of a cubic graph without leaves which cannot be split into three factors of the first degree, and which also cannot be drawn on a sphere without intersections [cf. the preceding review]. *D. C. Lewis* (Baltimore, Md.).

Sierpiński, Wacław. Sur un exemple de M. Kunugi de la théorie des espaces abstraits. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. **42** (1949), 4–8 (1952). (French. Polish summary)

Using the well-ordering theorem and the fact that the initial number of $Z(2^{\aleph_0})$ is not cofinal with any enumerable sequence of smaller ordinal numbers, Kunugui [J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. **4**, 1–40 (1936), pp. 9–13] gave an example of a $(\mathfrak{v})$ space of Fréchet [Les espaces abstraits, Gauthier-Villars, Paris, 1928, p. 172], X, a compact metric space Y, and a closed set M in $X\times Y$, whose projection on X is not analytic in X. The present note contains an effective example of this sort, whose demonstration avoids the use of transfinite numbers. *F. Bagemihl* (Rochester, N. Y.).

Inagaki, Takeshi. Contribution à la topologie. I. Math. J. Okayama Univ. **1**, 129–166 (1952).

In this paper the author extends work of G. Birkhoff [Ann. of Math. **38**, 39–56 (1937)], J. W. Tukey [Convergence and uniformity in topology, Princeton, 1940; these Rev. **2**, 67], and the reviewer [Duke Math. J. **11**, 181–199 (1944); these Rev. **5**, 212] on the relationships among convergence, closure, and neighborhood topologies in a space X. The new generality is obtained by consideration of topologies in every subspace H of X; that is, to each pair M, H of subsets of X is given a closure $\bar{M}^H$ of M relative to H. §1 discusses ordered systems in which the relation is reflexive and transitive. It collects a number of known results (Th. 1–10) and gives two new conditions equivalent to Tukey's relation $A\geqq B$. §2 describes the standard transformations to be used from one topology to another, shows the equivalence of different methods under a restriction on closure called "generalized monotony" ($M\subset N$ implies $\bar{M}^H\subset\bar{N}^H$ if M, N, and $H\subset X$), and then carefully gives many of the special conditions in all three methods equivalent to usual restrictions, such as additivity of closure, on any one of them. §3 considers convergence of ordered systems of non-empty subsets of X; §4 considers embeddability of X in a compact space. *M. M. Day* (Urbana, Ill.).

Shirota, Taira. On spaces with a complete structure. Proc. Japan Acad. **27**, 513–516 (1951).

The following theorem, as well as some eight preparatory results, is stated. Let X be a space whose cardinal number is weakly accessible in the sense of Tarski [Fund. Math. **30**, 68–89, 132–155 (1938)]. Then the following are equivalent: a) X is homeomorphic to a closed set of a topological product of the reals; b) there exists a complete uniform structure for X; c) X is complete in the uniformity [J. W. Tukey, Convergence and uniformity in topology, Princeton Univ. Press, 1940; these Rev. **2**, 67] consisting of all countable normal coverings. Proofs are not given. *R. Arens.*

Morita, Kiiti. On the simple extension of a space with respect to a uniformity. I. Proc. Japan Acad. **27**, 65–72 (1951).

The problem is that of imbedding a uniform space R in a complete space. The space R is assumed to be topologized by a closure which is not required to be additive. Suppose that R has a uniformity $[\mathfrak{U}_\alpha|\alpha\in\Omega]$ where $\mathfrak{U}_\alpha$ is an open covering of R for each $\alpha\in\Omega$. If every pair $\mathfrak{U}_\alpha$, $\mathfrak{U}_\beta$ has a refinement $\mathfrak{U}_\gamma$, α, β, $\gamma\in\Omega$, $[\mathfrak{U}_\alpha]$ is called a T-uniformity. If for $\alpha\in\Omega$ there is $\lambda(\alpha)\in\Omega$ such that for $U\in\mathfrak{U}_{\lambda(\alpha)}$ there are $U_\alpha\in\mathfrak{U}_\alpha$ and $\delta=\delta(\alpha,\lambda(\alpha))\in\Omega$ such that $S(U,\mathfrak{U}_\delta)\subset U_\alpha$, then $[\mathfrak{U}_\alpha]$ is called a regular uniformity. (Here $S(A,\mathfrak{U})$ is the union of all $U\in\mathfrak{U}$ which intersect A.) If for $\alpha\in\Omega$ there is $\lambda(\alpha)\in\Omega$ such that if $U\in\mathfrak{U}_{\lambda(\alpha)}$, there is $U_\alpha\in\mathfrak{U}_\alpha$ for which $S(U,\mathfrak{U}_{\lambda(\alpha)})\subset U_\alpha$, then $[\mathfrak{U}_\alpha]$ is called a completely regular uniformity. A uniform space in the sense of A. Weil or J. W. Tukey is a T_1-space with a completely regular T-uniformity which agrees with its topology. A T_1-space with a regular T-uniformity agreeing with its topology is a uniform space which has been considered by the reviewer. A space R has a uniformity, a regular uniformity, a completely regular uniformity, or a T-uniformity agreeing with its topology if and only if R is a weakly regular, regular, completely regular, or weakly regular T-space (i.e., closure is additive) respectively.

A family $[X_\lambda|\lambda\in\Lambda]$ of subsets of a space R having a uniformity $[\mathfrak{U}_\alpha]$ is called a Cauchy family if it has the finite intersection property and if for $\alpha\in\Omega$ there are $X_\lambda\in[X_\lambda]$, $\beta\in\Omega$, $U_\alpha\in\mathfrak{U}_\alpha$ such that $S(X_\lambda,\mathfrak{U}_\beta)\subset U_\alpha$. If $\bigcap_{\lambda\in\Lambda}\bar{X}_\lambda=0$, the family $[X_\lambda]$ is said to vanish. R is called complete with respect to the uniformity $[\mathfrak{U}_\alpha]$ if no Cauchy family vanishes. An equivalence relation for Cauchy families is defined relative to $[\mathfrak{U}_\alpha]$. If $[\mathfrak{U}_\alpha]$ agrees with the topology of R, either all or none of a class of equivalent Cauchy families vanishes. The simple extension R^*, whose completeness is studied, of R is defined as follows: An element of R^* is an element of R or an equivalence class of vanishing Cauchy families; an open set G^* is the union of a set G open in R and the set of classes of vanishing Cauchy families $[X_\lambda]$ such that for some $X_\lambda\in[X_\lambda]$ and $\alpha\in\Omega$, $S(X_\lambda,\mathfrak{U}_\alpha)\subset G$. The set of coverings $[\mathfrak{U}_\alpha^*|\alpha\in\Omega]$ is a uniformity for R^* with the same regu-

larity properties as $[\mathfrak{U}_\alpha]$ has for R. Topological properties of R are carried over to R^*. It is shown that if $[\mathfrak{U}_\alpha]$ is a (completely) regular uniformity which agrees with the topology of R, then R^* is complete with respect to $[\mathfrak{U}_\alpha^*]$. This theorem contains results obtained by A. Weil [Sur les espaces à structure uniforme et sur la topologie générale, Hermann, Paris, 1938] and by the reviewer [Duke Math. J. 5, 174–183 (1939)]. *L. W. Cohen* (Princeton, N. J.).

Morita, Kiiti. On the simple extension of a space with respect to a uniformity. II. Proc. Japan Acad. 27, 130–137 (1951).

This note studies the simple extension R^* [see the preceding review] of a space R with a uniformity $[\mathfrak{U}_\alpha | \alpha \in \Omega]$ as a means of compactification. It is shown that the simple extension R^* with respect to the uniformity consisting of all finite coverings of R is bicompact. If S is a bicompact extension of R such that the sets $S-\overline{R-G}$, the bar indicating closure in S, form a basis for the open sets in S when G varies over a basis for the open sets in R, the points of $S-R$ are closed in S, and $\bigcup_{i=1}^{n} G_i = R$ implies $\bigcup_{i=1}^{n}[S-\overline{R-G_i}] = S$, then S is homeomorphic with R^* by a mapping which leaves the points of R fixed. (Here R^* is the simple extension of R with respect to the uniformity of all finite coverings of R.) This result yields Shanin's theory [C. R. (Doklady) Acad. Sci. URSS 38, 154–156 (1943); these Rev. 5, 46] under appropriate restrictions. A theorem on the extension of uniformly continuous functions yields the Čech bicompactification and there are connections with analogous results of Wallman and Tukey. *L. W. Cohen.*

Morita, Kiiti. On the simple extension of a space with respect to a uniformity. III. Proc. Japan Acad. 27, 166–171 (1951).

It was shown in a previous note [see the two preceding reviews] that the simple extension R^* of a space R with a uniformity $[\mathfrak{U}_\alpha]$ has a uniformity $[\mathfrak{U}_\alpha^*]$ with respect to which R^* is complete whenever $[\mathfrak{U}_\alpha]$ is regular and agrees with the topology of R. In general R^* is not complete. The author iterates the operation of simple extension transfinitely and arrives at a space $\bar{R}$ having a uniformity with respect to which it is complete. Invariant properties of $\bar{R}$ are studied and theorems on continuous functions, metric properties, and bicompactification are obtained.

L. W. Cohen (Princeton, N. J.).

Suzuki, Jingoro. On the metrization and the completion of a space with respect to a uniformity. Proc. Japan Acad. 27, 219–223 (1951).

It is shown that if a countable number of open coverings of a T_1-space R forms a regular T-uniformity in the sense of Morita [see the third preceding review], then R is metrizable. This answers in the affirmative a question put by the reviewer [Duke Math. J. 5, 174–183 (1939)]. Dealing with the completion problem the author defines a Cauchy family of sets in a more restricted manner than does Morita [loc. cit.]. A class $[P_\lambda | \lambda \in \Lambda]$ of subsets of a space R with a uniformity $[\mathfrak{U}_\alpha | \alpha \in \Omega]$ is a Cauchy family if it has the finite intersection property and if for $\alpha \in \Omega$ there are $P_\lambda \in [P_\lambda]$, $\beta \in \Omega$, and $U_\alpha \in \mathfrak{U}_\alpha$ such that $S^2(P_\lambda, \mathfrak{U}_\beta) \subset U_\alpha$. The extension R^+ of R which results from the adjunction of the equivalence classes of vanishing Cauchy families to R is defined in the manner of Morita as are open sets in R^+ and a uniformity $[\mathfrak{U}_\alpha^+]$ for R^+. It is shown that R^+ is complete with respect to $[\mathfrak{U}_\alpha^+]$. In general R^+ is a proper subspace of the simple extension R^* but the two extensions coincide if $[\mathfrak{U}_\alpha]$ is a regular T-uniformity. *L. W. Cohen* (Princeton, N. J.).

Jaffard, Paul. Sur la complétion d'un espace uniforme quelconque. C. R. Acad. Sci. Paris 234, 502–503 (1952).

Nous utiliserons dans ce rapport la terminologie de notre livre [Appert et Fan, Espaces topologiques intermédiaires . . . , Hermann, Paris, 1951, pp. 120–135, spécial pp. 131–135; ces Rev. 13, 54], terminologie différente de la terminologie bourbakiste employée par l'auteur. A. Weil [Sur les espaces à structure uniforme . . . , Hermann, Paris, 1938, p. 19] avait démontré qu'étant donné un espace uniforme P, il existe un espace uniforme complet $\hat{P}$, unique à une isomorphie près, tel que P soit partout dense dans $\hat{P}$. L'espace $\hat{P}$ est appelé le complété de P. Bourbaki [Eléments de mathématique, Partie I, Livre III, Topologie générale, ch. II, p. 102, Hermann, Paris, 1940; ces Rev. 3, 55] avait généralisé ce résultat en démontrant qu'étant donné un espace quasi-uniforme E, il existe un espace quasi-uniforme complet $\hat{E}$ tel que E soit partout dense dans $\hat{E}$. L'auteur précise ce dernier théorème en démontrant que l'on peut toujours s'arranger pour que $\hat{E}-E$ satisfasse à l'axiome de séparation de Hausdorff, autrement dit, soit un espace uniforme; et que, lorsque l'on impose cette condition supplémentaire, $\hat{E}$ est unique à une isomorphie près. De plus $\hat{E}$ supposé tel que $\hat{E}-E$ satisfasse à l'axiome de séparation de Hausdorff, jouit de la propriété que voici: Si E' est un espace quasi-uniforme complet contenant E, alors la transformation identique de E en lui-même se prolonge en une isomorphie de $\hat{E}$ en un sous-espace E'' de E'.

A. Appert (Angers).

Dixmier, J. Sur certains espaces considérés par M. H. Stone. Summa Brasil. Math. 2, 151–182 (1951).

A compact space E is called stonian if the totality of continuous functions $f(x)$ on E constitutes a complete lattice with the usual semi-order [cf. M. H. Stone, Canadian J. Math. 1, 176–186 (1949); these Rev. 10, 546]. A stonian space E is called hyperstonian if the union of the carriers (supports) of the positive normal measures on E is dense in E. Here a countably additive (for Baire sets) measure μ is called normal if $\int_E f(x)\mu(dx) = \sup_i \int_E f_i(x)\mu(dx)$ when $f = \sup_i f_i$ of a bounded set $\{f_i\}$ of continuous functions with the property: for any pair (f_k, f_l) from $\{f_i\}$, there exists an f in $\{f_i\}$ such that $f \geqq f_k, f_l$. Proposition 9 states that a stonian space E is uniquely decomposable into three parts E_1, E_2, and E_3 such that: (i) There exists in E_1 a dense set of first category and every normal measure in E_1 is 0; (ii) E_2 is hyperstonian; (iii) in E_3 every set of first category is non-dense and the carrier of every measure is non-dense. Two characterisations of the hyperstonian spaces are given: firstly as the spectrum [Gelfand and Raikov, Mat. Sbornik 13(55), 301–316 (1943); cf. the reviewer, Proc. Imp. Acad. Tokyo 19, 356–359 (1943); these Rev. 6, 147; 7, 253] of the self-adjoint algebra $L^\infty(R, \mu)$ of all the essentially bounded measurable (with respect to μ) functions on a locally compact space R, and secondly as the spectrum of a self-adjoint algebra of operators on a properly chosen Hilbert space. Also, a stonian space is characterised as the spectrum of the totality of the self-adjoint algebra of the totality of bounded Borel functions (modulo those vanishing on a set of first category) on a Hausdorff space. In this way, an example is given of a stonian space which is not hyperstonian. *K. Yosida* (Nagoya).

Smirnov, Yu. M. On the metrization of topological spaces. Uspehi Matem. Nauk (N.S.) 6, no. 6(46), 100–111 (1951). (Russian)

A survey of the metrization problem, beginning with Uryson's theorems, discussing the unsatisfactory results of Aleksandrov-Uryson, Chittenden, Tukey, and Bing, and exposing the author's recently published solution [Doklady Akad. Nauk SSSR 77, 197–200 (1951); these Rev. 12, 845]. A. H. Stone's theorem that every metrizable space is paracompact and Dieudonné's theorem that every paracompact Hausdorff space is normal are proved by very brief arguments. It is shown also that a locally metrizable Hausdorff space is metrizable if and only if it is paracompact. [Reviewer's note: In this theorem, no separation axiom need be assumed.] *E. Hewitt* (Seattle, Wash.).

Smirnov, Yu. M. On the weight of the ring of bounded continuous functions over a normal space. Mat. Sbornik N.S. 30(72), 213–218 (1952). (Russian)

Let R be a normal space. Let $\sigma(R)$ be the combinatorial weight (character) of R defined earlier by the author [Doklady Akad. Nauk SSSR 69, 611–613 (1949); these Rev. 11, 381]. Let $\pi(R)$ be the least cardinal number of a family $\mathfrak{A}$ of open subsets of R such that for every closed set $F \subset R$ and open set $G \supset F$, there exists an $A \in \mathfrak{A}$ such that $F \subset A \subset G$. Let τX, for any topological space X, be the least cardinal number of an open basis for X. Let the ring $C(R)$ of all bounded real-valued continuous functions on R be topologized by the uniform metric. Then $\sigma(R) = \pi(R) = \tau(C(R))$. If R is compact, then $\tau(R) = \tau(C(R))$. Thus $\tau(\beta R) = \sigma(R)$. *E. Hewitt* (Seattle, Wash.).

Vulih, B. Z. On the extension of continuous functions in topological spaces. Mat. Sbornik N.S. 30(72), 167–170 (1952). (Russian)

Let S be a topological space containing a dense subset A. Then every continuous real-valued function on A admits a continuous extension over S if and only if every pair of completely separated subsets of A have disjoint closures in S. An equivalent result has been proved by the reviewer [Anais Acad. Brasil. de Ci. 21, 175–179 (1949); these Rev. 11, 194]. *E. Hewitt* (Seattle, Wash.).

Kuratowski, C. Sur quelques problèmes topologiques concernant le prolongement des fonctions continues. Colloquium Math. 2 (1951), 186–191 (1952).

The concepts are those of the author's book [Topologie, v. II, Warszawa-Wrocław, 1950; these Rev. 12, 517]. The material (mainly taken from the book) is presumably introductory to a set of four interesting problems following the paper. Write $X \tau Y$ if, when A is closed in X and $f \in Y^A$, there is an extension $f^* \in Y^X$ of f. One new result asserts that (among compacta and with Y an ANR) the relation X non-τ Y is invariant under "petites transformations". Two of the problems follow: If H is Hilbert's cube, does $H \tau Y$ imply that Y is an AR? Do the conditions $X \tau Y$ and $\dim X = n$ imply $I^n \tau Y$? Special cases are known. *A. D. Wallace* (New Orleans, La.).

Sikorski, Roman. Dimension theory in closure algebras. Fund. Math. 38, 153–166 (1951).

This paper is a continuation of an earlier one [Sikorski, Fund. Math. 36, 165–206 (1949); these Rev. 12, 85]. Dimension is defined inductively in terms of separability of closed elements by open ones. The paper is not devoted to a mere characterization of those theorems of dimension theory which hold for closure algebras; and there appears to be no difficulty in extending results which hold for separable metric spaces to C-algebras [see the paper cited, or its review]. It is proved that the compact n-dimensional space universal for all n-dimensional spaces is also universal for all n-dimensional C-algebras. In connection with the problem of deducing "topological" properties of A/I from those of A and I (I being a σ-ideal of the closure algebra A), it is proved that $\dim A/I \leqq \dim A$. *R. Arens.*

Miyazaki, Hiroshi. A note on paracompact spaces. Tôhoku Math. J. (2) 4, 88–92 (1952).

Extending a result of Begle [Bull. Amer. Math. Soc. 55, 577–579 (1949); these Rev. 10, 726], the author proves that if A is a paracompact normal space and B a compact normal space, then $A \times B$ is paracompact and normal, and $\dim (A \times B) \leqq \dim A + \dim B$. Here "dim" denotes the covering dimension [see Dowker, Amer. J. Math. 69, 200–242 (1947); these Rev. 8, 594]. This theorem is deduced from the special case of it in which A is a simplicial complex with the weak topology, and this special case is established by an argument using mappings in spheres. [On p. 90, lines 17–19 and p. 91, line 4*, "n" should be "m", and in the proof (p. 90) that $\dim K \leqq \dim^* K$, "S^n" should be "S^m" throughout.] *A. H. Stone* (Manchester).

Bourgin, D. G. The paracompactness of the weak simplicial complex. Proc. Nat. Acad. Sci. U. S. A. 38, 305–313 (1952).

The author shows that a simplicial complex with the weak topology (a set is open if the intersection with every closed simplex is relatively open) is a paracompact T_2-space. The proof goes by induction over the skeletons of the complex, and is somewhat too complicated to be described here. It is also proved that the weak topology is equivalent to the barycentric topology; the latter is defined by choosing for each vertex a unit interval and embedding the complex into the product of all these intervals by means of barycentric coordinates. Several related results are given. *H. Samelson* (Princeton, N. J.).

Łoś, J., and Ryll-Nardzewski, C. On the application of Tychonoff's theorem in mathematical proofs. Fund. Math. 38, 233–237 (1951).

This paper centers around the use of the Tychonoff product theorem (the product of compact spaces is compact) as a substitute for the axiom of choice. (Recently it has been shown that the theorem implies the choice axiom [J. L. Kelley, Fund. Math. 37, 75–76 (1950); these Rev. 12, 626].) First, without use of the choice axiom (or the Tychonoff theorem) the authors demonstrate: (1) Let Y be a compact topological space, let A be a set and for each $y \in Y$ let P_y be a property of finite character (i.e., a property which is possessed by a set if and only if it is possessed by every finite subset). If for each finite subset E of A the set of all $y \in Y$ such that E has P_y is a closed non-void subset of Y, then for some $y \in Y$ it is true that A has P_y.

Using the Tychonoff theorem the following proposition is then proved: (2) For each member a of a set A let Y_a be a compact topological space, and let R be a symmetric relation satisfying the condition: for each a and b in A the set of all $(x, y) \in Y_a \times Y_b$ such that xRy is closed in $Y_a \times Y_b$. A consistent choice for a subset E of A is defined to be a function f on E such that $f(a) \in Y_a$ and $f(a) R f(b)$ for all a and b in A. Then there is a consistent choice for A if there is a consistent choice for each finite subset of A.

As applications, (1) together with the Tychonoff theorem is used to prove that each partial ordering has an extension which is a simple ordering, and (2) is used to prove the Hahn-Banach theorem on the extension of linear functionals. (A proof of the latter by means of the Tychonoff theorem was constructed by S. Kakutani, who then suggested to the reviewer the theorem cited at the beginning of the review.)

J. L. Kelley (Lawrence, Kan.).

Estill, Mary Ellen. A primitive dispersion set of the plane. Duke Math. J. **19**, 323–328 (1952).

This paper exhibits a primitive dispersion set of a simple closed curve plus its interior, and thus presents an affirmative solution to the first unsolved problem listed by R. L. Wilder in his book "Topology of manifolds" [Amer. Math. Soc. Colloq. Publ., v. 32, New York, 1949; these Rev. **10**, 614].

D. W. Hall (College Park, Md.).

Jones, F. Burton. Concerning aposyndetic and non-aposyndetic continua. Bull. Amer. Math. Soc. **58**, 137–151 (1952).

The author gives a highly readable expository survey of a theory of non-locally connected subcontinua of the 2-sphere, a theory due primarily to Wilder, Whyburn, and himself. Ten figures are given.

A (compact) continuum M is aposyndetic at a point p of M if for each point k of $M-p$ there is an open subset U of M, and a continuum H, such that $p \in U \subset H \subset M-k$ [Jones, Amer. J. Math. **63**, 545–553 (1941); these Rev. **3**, 59]. M is semi-locally connected at $p \in M$ if every neighborhood U of p contains a neighborhood V of p such that $M-U$ lies in the union of a finite number of components of $M-V$ [Whyburn, ibid. **61**, 733–749 (1939); these Rev. **1**, 31]. M is (everywhere) aposyndetic if and only if M is (everywhere) semi-locally connected, but pointwise neither property implies the other. Both are closely related to r-avoidability (for $r=0$) in the sense of Wilder [Časopis Pěst. Mat. Fys. **67**, 185–197 (1938)]. Following is a sample, rather than a summary, of the results cited. S^2 denotes the 2-sphere. (I) (Jones). If M is a continuum in S^2, and D is a component of S^2-M, then S^2-D is connected im kleinen at every point where M is aposyndetic. (M is connected im kleinen at p if every neighborhood U of p contains a neighborhood V of p which lies in a single component of U.) (II) (Wilder-Whyburn). If M is an aposyndetic continuum in S^2, then the boundary of every component of S^2-M is locally connected (a generalization of the Torhorst Theorem, in which M is required to be locally connected). (III) (Jones). For a continuum $M \subset S^2$ to be aposyndetic it is necessary and sufficient that S^2-M be non-folded. (For definition see Jones [Bull. Amer. Math. Soc. **53**, 170–175 (1947); these Rev. **8**, 397].) (IV) (Whyburn). If M is aposyndetic, and $p \in M$, then $M-p$ is continuum-wise connected if and only if $M-p$ is connected. (V) (Whyburn-Jones). The Cartesian product of any two nondegenerate continua is aposyndetic. The author closes with a discussion of the problem of arranging continua in a "spectrum", in the order of the simplicity and regularity of their structure.

E. E. Moise (Ann Arbor, Mich.).

Anderson, R. D. Monotone interior dimension-raising mappings. Duke Math. J. **19**, 359–366 (1952).

The author shows that there is a monotone interior map of a one-dimensional compact continuum onto the fundamental cube in Hilbert space. The construction proceeds directly, in three-dimensional euclidean space E_3, and some of the preliminary results are of independent interest. This result definitely disposes of an old problem. Hurewicz [Fund. Math. **15**, 57–60 (1930)] showed that there is a monotone map of a one-dimensional set onto the Hilbert cube; Kolmogoroff proved that there is an interior map of a compact space which raises dimension [Ann. of Math. **38**, 36–38 (1937)]; and Bing's recent proof of the existence of hereditarily indecomposable continua of all dimensions [Trans. Amer. Math. Soc. **71**, 267–273 (1951); these Rev. **13**, 265] together with a result of the reviewer [ibid. **52**, 22–36 (1942); these Rev. **3**, 315] implies that there is a monotone interior map of a two-dimensional compact continuum in E_3 onto an infinite-dimensional space. The author's theorem closes the question.

J. L. Kelley.

***Keldyš, L. V. Monotone mappings of the cube.** Trudy Mat. Inst. Steklov., v. 38, pp. 72–76. Izdat. Akad. Nauk SSSR, Moscow, 1951. (Russian) 20 rubles.

First, an example is constructed of a one-dimensional continuum X and a monotone mapping of X onto the Hilbert parallelepiped. Secondly, this example is used to construct an irreducible monotone mapping f of the 3-dimensional cube C_3 onto an infinite-dimensional space, so that if U is open in C_3 then $f(U)$ contains a homeomorph of the Hilbert parallelepiped. [Reviewer's comments. Several references should be added. In connection with the first example, the earliest example of a monotone map of a one-dimensional curve onto a square seems to be due to R. L. Moore. The statement of this example is implicit in a remark of Menger [Fund. Math. **10**, 96–115 (1927), p. 107]. Later, Hurewicz [ibid. **15**, 57–60 (1930)] and Mazurkiewicz [C. R. Congrès Math. Pays Slaves, 1929, Warsaw, 1930, pp. 66–71] independently proved that every continuum Y is the monotone image of some one-dimensional continuum X. Moreover, Anderson [Trans. Amer. Math. Soc. **67**, 451–460 (1949); these Rev. **11**, 453] proved that if Y is locally connected, then X may be chosen so as to be locally connected. In connection with the second example, Anderson [Bull. Amer. Math. Soc. **58**, 393 (1952)] has quite recently obtained a similar example but considerably stronger in that the monotone mapping f is also interior.

E. E. Floyd.

Sikorski, R. Homomorphisms, mappings and retracts. Colloquium Math. **2** (1951), 202–211 (1952).

The main purpose of this paper is a characterization of fields $\mathfrak{Y}$ of subsets of a set $\mathbf{Y}$ that have the following property: Whenever there is a homomorphism (i.e., a mapping which preserves finite unions and complements) h of $\mathfrak{Y}$ into a quotient algebra $\mathfrak{X}/\mathfrak{s}$ (where $\mathfrak{s}$ is an ideal in a field $\mathfrak{X}$ of subsets of a set $\mathbf{X}$), then there is a point mapping φ of $\mathbf{X}$ into $\mathbf{Y}$ such that $Y \in \mathfrak{Y}$ implies $\varphi^{-1}(Y) \in h(Y)$. This property is shown to be equivalent to the following one: In M. H. Stone's representation of $\mathfrak{Y}$ as an isomorphic image of the field $\mathfrak{K}$ of all open-closed subsets of a totally disconnected bicompact space $\mathbf{S}$ the isomorphism can be extended to the σ-fields generated by $\mathfrak{Y}$ and $\mathfrak{K}$ and $\mathbf{S}$ is a retract of every totally disconnected bicompact space containing it.

H. M. Schaerf (St. Louis, Mo.).

Borsuk, K. An example of a finite dimensional continuum having an infinite number of Cartesian factors. Colloquium Math. **2** (1951), 192–193 (1952).

The space A is a Cartesian factor of M if for some B the spaces $A \times B$ and M are homeomorphic. If a space has no other than one-point Cartesian factors, then it is called prime. An example is given of an AR in R^3 (of dimension 3)

having infinitely many topologically different prime Cartesian factors. *A. D. Wallace* (New Orleans, La.).

Gindifer, Mieczysław. On generalized spheres. Fund. Math. **38**, 167–178 (1951).

Let A be a non-empty subset of a metric space S. By a generalized sphere $K_r(A, S)$ with center A and radius r is meant the set of all points of S whose distance from A does not exceed r. For the applications the author confines himself to the case where S is the n-dimensional Euclidean space. Every generalized sphere with positive radius and compact center is shown to be locally connected. A positive number r is said to be a singular radius for the subset A of S provided there exists a point p in $K_r(A, S)$ such that this set is not locally contractible in p. The paper demonstrates the existence of compact subsets of the Euclidean plane whose sets of singular radii have the power of the continuum. The following interesting problem is proposed. Let A be a compact subset of the n-dimensional Euclidean space S. Let R denote the set of all positive numbers r such that $K_r(A, S)$ is not homeomorphic to a polytope. The problem is whether the set R is necessarily of first category (in the sense of Baire) and of measure zero (in the sense of Lebesgue). *D. W. Hall* (College Park, Md.).

Wang, Hsien-Chung. A remark on transformation groups leaving fixed an end point. Proc. Amer. Math. Soc. **3**, 548–549 (1952).

This note was motivated by the following problem of A. D. Wallace [Fund. Math. **36**, 119–124 (1949); these Rev. **11**, 734]: Let G be a compact transformation group of a Peano continuum X leaving fixed an end point of X. Must G have other fixed points? If G is cyclic instead of compact, the answer to this question is known to be in the affirmative [see Wallace, Bull. Amer. Math. Soc. **51**, 413–416 (1945); these Rev. **6**, 278]. The present author solves the problem here stated for spaces more general than Peano continua by proving the following theorem: Let G be a transformation group of an arcwise connected Hausdorff space X leaving fixed an end point e of X. Then G has no other fixed point if and only if, given any neighborhood U of e, the orbit $G(U)$ under G coincides with the whole space X. If G, X, e have the same meaning as in the theorem and G is, moreover, compact, then G has a fixed point distinct from e.

D. W. Hall (College Park, Md.).

Forrester, Amasa. A theorem on involutory transformations without fixed points. Proc. Amer. Math. Soc. **3**, 333–334 (1952).

Let ϕ be a topological automorphism of period 2 of a Euclidean n-sphere S. It is shown that if ϕ has no fixed points, the union of the linear segments $(Q, \phi Q)$, $Q \in S$, is the whole interior region of S. *P. A. Smith.*

Moise, Edwin E. Affine structures in 3-manifolds. V. The triangulation theorem and Hauptvermutung. Ann. of Math. (2) **56**, 96–114 (1952).

In the important case of the general topological 3-manifold, both the triangulation theorem and the so-called Hauptvermutung are established. In other words, it is shown that every 3-manifold is triangulable (into a combinatorial manifold) and that homeomorphic 3-manifolds are combinatorially equivalent. For 3-manifolds, these far-reaching results afford a unified basis for the combined development of set-theoretic and combinatorial topology.

S. S. Cairns (Urbana, Ill.).

Borsuk, Karol. Concerning the Cartesian product of Cantor-manifolds. Fund. Math. **38**, 55–72 (1951).

An n-dimensional Cantor-manifold is a compaction of dimension n which is not disconnected by any subset of dimension $\leqq n-2$. If A and B are polyhedra, then $C=A\times B$ is a Cantor-manifold if and only if both A and B are Cantor-manifolds. It is shown here that this statement is no longer correct when we move outside the class of polyhedra. Examples are presented to show that any two of A, B, and $C=A\times B$ can be Cantor-manifolds without the third being one, and also that C can be one without either of the factors being one. In each example all the sets are locally connected continua. The construction of these examples uses an example due to Pontrjagin which he used to disprove the hypothesis that $\dim(X\times Y)=\dim X+\dim Y$. A full discussion of this example is included in the present paper.

E. G. Begle (New Haven, Conn.).

White, Paul A. Some characterizations of generalized manifolds with boundaries. Canadian J. Math. **4**, 329–342 (1952).

Characterizations of generalized manifolds with boundary are given in terms of the boundary and its complement, and relations between the manifold and its boundary are studied. Both the unoriented and oriented cases are treated. Results are too numerous to list here. A similar study is made of the special case of the generalized n-cell. *E. G. Begle.*

Rohlin, V. A. A three-dimensional manifold is the boundary of a four-dimensional one. Doklady Akad. Nauk SSSR (N.S.) **81**, 355–357 (1951). (Russian)

The author sketches a proof that a compact, connected, orientable 3-manifold M^3 is homeomorphic to the boundary of some orientable 4-manifold M^4. (Manifolds and homeomorphisms are assumed to be "smooth," where "smooth" is not defined.) The indicated proof proceeds in three steps. (1) By a modification of an old construction due to Kneser, the proposition is correct if M^3 can be imbedded in 5-dimensional euclidean space R^5. Two operations O_1 and O_2 are defined: $O_1(M^3)$ is obtained from M^3 by boring out a tube and identifying the points of the bounding torus in pairs to make a Klein bottle, $O_2(M^3)$ is obtained from M^3 by boring out two tubes and matching the boundary tori with each other. Then (2) M^3 must bound if $O_i(M^3)$ does. Finally (3) an arbitrary M^3 may be transformed by a finite sequence of operations O_i into an M_1^3 that can be imbedded in R^5. It is stated that any compact, 3-manifold, orientable or not, is the boundary of some 4-manifold. (The announced results have been recently obtained also by R. Thom [Colloque de Topologie, Strassbourg, 1951, no. V] using a different method. Thom's paper also includes the generalization of (1) to higher dimensions.) *R. H. Fox* (Princeton, N. J.).

Krasnosel'skiĭ, M. A., and Krein, S. G. On a proof of the theorem on category of a projective space. Ukrain. Mat. Žurnal **1**, no. 2, 99–102 (1949). (Russian)

New proofs are given for the following classical theorems of Lusternik and Schnirelmann [Topological methods in variational problems, Moscow, 1930]: (1) if an n-sphere is covered by k closed subsets none of which contains a pair of antipodal points, then $k>n$; (2) the category of projective n-space is $n+1$. The proofs utilize a theorem which is in the nature of a generalization of Sperner's lemma.

E. E. Floyd (Charlottesville, Va.).

Jackson, James R. On homotopy groups of function spaces. Amer. J. Math. 74, 241–252 (1952).

The paper is concerned with the homotopy groups π_m, $m=1, 2, \cdots$, of the following spaces of continuous mappings: Ω of $(X, X_0)\to(Y, Y_0)$, Ω_0 of $(X, X_0)\to(Y, y_0)$, Ω_{00} of $(X, X_0, x_0)\to(Y, Y_0, y_0)$, Ψ of $X_0\to Y_0$, Ψ_0 of $(X_0, x_0)\to(Y_0, y_0)$, where X and Y are spaces satisfying some general conditions (e.g., X an ANR or X metric and Y an ANR), X_0 is a retract of X, Y_0 is a subset of Y, and x_0 and y_0 are points of X_0 and Y_0 respectively. The main result is that $\pi_m(\Omega, y_0)$ is a split extension of $\pi_m(\Omega_0, y_0)$ by $\pi_m(\Psi, y_0)\approx\pi_m(\Omega, \Omega_0, y_0)$ and that $\pi_m(\Omega_{00}, y_0)$ is a split extension of $\pi_m(\Omega_0, y_0)$ by $\pi_m(\Psi_0, y_0)\approx\pi_m(\Omega_{00}, \Omega_0, y_0)$. Furthermore, $\pi_m(\Omega_0, y_0)$ is trivial if X_0 is a deformation retract of X. As an application the author derives a generalization of the reviewer's structure theorem for torus homotopy groups. Among other applications given, the most interesting is to the closed orientable surface X_p of genus p: the group $\pi_m(Y^{X_p}, y_0)$ may be built up by a sequence of split extensions from $\pi_m(Y, y_0)$, $\pi_{m+2}(Y, y_0)$ and $2p$ copies of $\pi_{m+1}(Y, y_0)$. *R. H. Fox.*

Uehara, Hiroshi. Some remarks on relatively free homotopy. J. Math. Soc. Japan 2, 247–252 (1951).

The fundamental group σ of the function space

$$((S^{n-1}, S^{n-2})\to(Z, Y))$$

is a split extension of $\pi_n(Z, Y, *)$ by $\pi_1(Y, *)$. The operation of $\pi_1(Y, *)$ on $\pi_n(Z, Y, *)$ is represented in σ as a conjugation. Thus relative n-simplicity of the pair (Z, Y) means that the extension is trivial. *R. H. Fox.*

Uehara, Hiroshi. On a generalization of the Abe groups J. Math. Soc. Japan 2, 231–246 (1951).

The author's generalized Abe group $\sigma^{(r,n)}(Z)$ is the fundamental group (with base point $*$) of the function space $((S^{n-1}, K^{n-1})\to(Z, *))$, where K^{n-1} is any given subcomplex of S^{n-1}. This reduces to Hu's Abhomotopy group $\mathfrak{X}^{(r,n)}(Z)$ when K^{n-1} is the $(n-1)$-sphere S^{n-1} and to the Abe group $\mathfrak{X}^r(Z)=\mathfrak{X}^{(r,0)}(Z)$ when K^{n-1} is vacuous. Except in the trivial case $K^{n-1}=S^{n-1}$, the group $\sigma^{(r,n)}(Z)$ has a subgroup isomorphic to the r-dimensional homotopy group $\pi^r(Z)$. There are $\binom{r-1}{n}$ natural isomorphic imbeddings of $\mathfrak{X}^{(r,n)}$ in the torus homotopy group τ^r; if $n>0$, $\mathfrak{X}^{(r,n)}$ is isomorphic to the direct sum of π^r and π^{n+1}. The groups $\mathfrak{X}^n(S^{n-n}\to Z)$ and $\tau^{r-n}((S^n, x_0)\to(Z, *))$ are analyzed. The concept of r-simplicity in the sense of Eilenberg is generalized to the concept of r-simplicity in the sense of torus homotopy groups; Z is r-simple in the sense of torus homotopy groups if $\tau^r(Z)$ is abelian. *R. H. Fox* (Princeton, N. J.).

Miyazaki, Hiroshi. On covering homotopy theorems. Tôhoku Math. J. (2) 4, 80–87 (1952).

The first and second homotopy theorems on fibre bundles are stated and proved by Steenrod in his book, "The topology of fibre bundles" [Princeton Univ. Press, 1951; these Rev. 12, 522], for C_r-spaces, that is, locally compact normal spaces of which every open covering has a countable subcovering. In the present paper, special cases of these covering homotopy theorems are given under some other conditions. Let $\mathfrak{B}$, $\mathfrak{B}'$ be two fibre bundles having the same fibre Y, the same structural group G, and the base spaces X and X' respectively. In the first special case, the author assumes that G is totally disconnected and that X is locally arcwise connected. In the second special case, he assumes that X is completely regular and that the bundle space B' of the bundle $\mathfrak{B}'$ is compact. In both cases, let h_0: $\mathfrak{B}\to\mathfrak{B}'$ be a bundle map and let $\bar{h}$: $X\times I\to X'$ be a homotopy of the induced map $\bar{h}_0$: $X\to X'$. Then there exists a homotopy h: $\mathfrak{B}\times I\to\mathfrak{B}'$ of h_0 whose induced homotopy is $\bar{h}$. Stationarity of h is satisfied in the second case but not in first case. *S. T. Hu* (New Orleans, La.).

Liao, S. D. A theorem on periodic transformations of homology spheres. Ann. of Math. (2) 56, 68–83 (1952).

Let X be a finite-dimensional compact Hausdorff space which is a homology n-sphere over J_p (integers, modulo p, p a prime). Let T be a topological automorphism of X of period p. According to a theorem of the reviewer [Lefschetz, Algebraic topology, Amer. Math. Soc. Colloq. Publ., v. 27, New York, 1942, Appendix B; these Rev. 4, 84], the fixed-point set of T is a homology r-sphere over J_p ($-1\leqq r\leqq p$). If X is an ordinary r-sphere and T is analytic, it is easy to verify the proposition (dimensional parity) that $n-r$ is even or odd according as T preserves or reverses orientation. For the topological case the author shows that this proposition holds when all the integral cohomology groups of X, in the theorem above, are finitely generated. (For the orientation-preserving case it was proved by the reviewer [Ann. of Math. 46, 357–364 (1945); these Rev. 7, 136] that $n-r$ is even provided X satisfies conditions which appear to be considerably more restrictive than those under consideration.) Let 0 be the space of orbits of T and let $\chi(0, J)$, $\chi(0, R)$ be the Euler characteristics of 0 over J_p and R (R being the group of rationals). It is shown that the parity of $\chi(0, J_n)$ is the same as that of $n-r$ and that the parity of $\chi(0, R)$ is determined by the preservation or reversal of orientation. It is shown that the two characteristics are equal, which proves the theorem. The present work overlaps to some extent with a recent paper by Floyd [Trans. Amer. Math. Soc. 72, 138–147 (1952); these Rev. 13, 673]. *P. A. Smith* (New York, N. Y.).

Larguier, Everett. Homology bases with applications to local connectedness. Pacific J. Math. 2, 191–208 (1952).

A sufficient condition is given for a compact Hausdorff space M to be embeddable as a G_δ subset in a locally connected M' which consists of the union of M and a countable number of arcs. The class of spaces satisfying this condition contains all metric and some non-metric spaces. A necessary condition is given for the case of separable (=countable dense subset) spaces M. The proofs depend on a study of the homology groups of M considered as a vector space. A special decreasing sequence of linear subspaces is defined, which, under suitable conditions, leads to a special basis for the homology group. For the class of spaces mentioned above, these conditions are satisfied for the 0-dimensional group, and the resulting basis is used in the construction of M'. *E. G. Begle* (New Haven, Conn.).

Hawley, N. S. Complex fiber bundles. Proc. Nat. Acad. Sci. U. S. A. 38, 411–415 (1952).

After suitably defining the appropriate terms, the author proposes the general problem of classifying complex fiber bundles, that is, determining all the equivalence classes of complex bundles over a given "base space" (a complex manifold) with a given fiber F (another complex manifold) and with a given "group of the bundle" G. If F is a complex projective space and G its projective group, a "projective bundle" is obtained. Theorems are given concerning the existence of such bundles, and their equivalence classes in relation to certain complex homotopy classes. Regular line bundles over closed Riemann surfaces are briefly discussed. *S. S. Cairns* (Urbana, Ill.).

Gugenheim, V. K. A. M. Some theorems on piecewise linear embedding. Proc. Nat. Acad. Sci. U. S. A. **38**, 333–337 (1952).

This paper is primarily concerned with compact, connected, oriented and possibly boundaried, combinatorial manifolds M and the "congruence classes" of semilinear imbeddings of an M^q in the interior of an M^n. Two such imbeddings are called by the author congruent if one can be carried into the other by a semilinear orientation-preserving homeomorphism of M^n on itself. If a triangulation of M^n is chosen in which M^q appears as a subcomplex, then to each interior point x of M^q is associated the imbedding of the $(q-1)$-sphere $Lk(x, M^q)$ in the $(n-1)$-sphere $Lk(x, M^n)$, where $Lk(x, M)$ means the boundary of the star of x in the given triangulation of M. The resulting congruence class of imbeddings of S^{q-1} in S^{n-1} is independent of the choice of subdivision (at least if $q<n-1$) and is denoted by $\lambda(x, M^q)$. If $n=2q$, $\lambda(x, M^q)=0$ for almost all x, i.e., the set X of points x at which the imbedding $\lambda(x, M^q)$ is unusual is finite. The principal result is that the congruence type of an imbedding of a q-dimensional element E^q in a $2q$-dimensional sphere S^{2q} is uniquely determined by the finite collection of congruence types $\lambda(x, E^q)$, $x \in X$. There are some auxiliary results of independent interest, of which only the most striking need be mentioned here. In a manner similar to the composition of knots [H. Schubert, S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. 1949, no. 3, 57–104; these Rev. 11, 196] the congruence types of an S^q in an S^n, $q<n$, may be "added" and these types then form a commutative semigroup (q, n). The congruence types of an E^q in an S^n form a commutative semigroup $[q, n]$ which is a homomorph of (q, n); $(q, n)=0$ if $q+1=n \leqq 3$ or if $2q+2 \leqq n$; $[q, n]=0$ if $q=1$ or if $q+1=n \leqq 4$ or if $2q+1 \leqq n$. The paper does not contain proofs.

R. H. Fox (Princeton, N. J.).

Radó, Tibor. Sulla teoria delle omologie singolari. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **8** (1950–51), 57–63 (1952). (English. Italian summary)

This paper is a summary of results which have since appeared elsewhere [Pacific J. Math. 1, 265–290 (1951); these Rev. 13, 373]. *J. L. Kelley* (Lawrence, Kan.).

Wada, Hidekazu. On mappings from complexes into the complex projective space. Tôhoku Math. J. (2) **4**, 69–76 (1952).

Let $P(n)$ denote the n-dimensional complex projective space, and K a polyhedron whose dimension is $\leqq 2n+1$. The main object of this paper is to consider the problem of classifying continuous maps of K into $P(n)$. The main result obtained may be stated as follows: Let $f, g: K \to P(n)$ be two maps which agree on the 2-skeleton of K. There is then defined a $(2n+1)$-dimensional cohomology class with integral coefficients, $d^{2n+1}(f, g)$, the so-called obstruction to a homotopy of f into g. Then f and g are homotopic if and only if there exists a 1-dimensional integral cohomology class w such that $d^{2n+1}(f, g)=2^n \cdot w \cdot (f^* s^2)$. Here s^2 is the fundamental 2-dimensional cohomology class of $P(n)$, and the product on the right is the ordinary cup-product of integral cohomology classes. The proof of this result is modelled after a paper of Steenrod [Ann. of Math. **48**, 290–320 (1947); these Rev. **9**, 154]. This problem has also been considered by Eckmann [cf. C. R. Acad. Sci. Paris **228**, 1397–1399 (1949); these Rev. **10**, 617]. The author also proves that if K is a polyhedron of arbitrary dimension whose first cohomology group vanishes, then the homotopy classes of maps of K into $P(n)$ are in 1-1 correspondence with the homotopy classes of maps of K into the $(2n+1)$-dimensional sphere. The proof makes use of the fibering of the $(2n+1)$-sphere by circles. *W. S. Massey.*

Bott, Raoul. Two new combinatorial invariants for polyhedra. Portugaliae Math. **11**, 35–40 (1952).

Let $b(K)$ be the nth Betti number of a finite n-complex K having n-cells $\sigma_1, \cdots, \sigma_N$. Let $\mathfrak{N}$ be the set of all subsets of the first N integers, $|s|$ the "length" of $s \in \mathfrak{N}$, and $K_s=K-\bigcup_{i \in s}\sigma_i$. The author proves the polynomials $R(K, \lambda)=\sum_{s \in \mathfrak{N}}(-1)^{|s|}\lambda^{b(K_s)}$ and $S(K, \lambda)=\sum_{s \in \mathfrak{N}} i^{c(K_s)}\lambda^{b(K_s)}$ ($i=\sqrt{-1}$, $c(K_s)=|s|+((n-1)-$Betti number of K_s) are invariant under subdivision and are not invariants of homotopy type. *J. Dugundji* (Princeton, N. J.).

Dugundji, J. Note on CW polytopes. Portugaliae Math. **11**, 7–10 (1952).

It is shown here that an arbitrary polyhedron, P, not necessarily locally finite, with the CW topology of J. H. C. Whitehead is paracompact and also that any map into P of a closed subset A of a metric space X has an extension to a neighborhood W of A in X. These results were known previously only for the locally finite case and, for the second result, separable X. It is also shown that this second result holds if X is a CW polyhedron. As a dual result, it is shown that if Y is a metric ANR, then any map into Y of a closed subset Q of a CW polytope P has an extension to a neighborhood W of Q in P. *E. G. Begle* (New Haven, Conn.).

Boltyanskiĭ, V. Vector fields on a manifold. Doklady Akad. Nauk SSSR (N.S.) **80**, 305–307 (1951). (Russian)

The author determines the range of the second obstruction $Z^{r+2}(\mathfrak{S}_k)$ against existence of a system $\mathfrak{S}_k$ of k independent vector fields in an orientable triangulable n-manifold M_n, assuming that the Stiefel class Y^{r+1} vanishes ($r=n-k$). The formula reads ($\mathfrak{S}_k^0$ is any specific k-field on the $(r+1)$-skeleton):

$$Z^{r+2}(\mathfrak{S}_k)=Z^{r+2}(\mathfrak{S}_k^0)+\bar{Y}_2(\mathfrak{S}_k^0) \cup D^r+\mathrm{Sq}_{r-2}D^r;$$

here D^r runs over $H^r(M_n, \pi_r(V_{n,k}))$; $\bar{Y}_2(\mathfrak{S}_k^0)$ is the obstruction against extending $\mathfrak{S}_k^0$ to an n-field over the 2-skeleton—for $r>2$ it reduces to Y^2; the coefficients are suitably paired. The proof utilizes the intersection behavior of two different barycentric subdivisions of the $(r+2)$-skeleton (one based on equal masses, the other on pairwise different masses at the vertices), which has been studied by Glezerman and Pontryagin [Uspehi Matem. Nauk **2**, no. 1(17), 58–155 (1947); these Rev. **10**, 391]. An $\mathfrak{S}_k$ is constructed, with arbitrarily prescribed difference cocycle $D^r(\mathfrak{S}_k, \mathfrak{S}_k^0)$, having singularities, in the interiors of $(r+2)$-simplices only, of three types, corresponding to the three terms in the formula. The second and third term are treated by referring to Pontryagin [C. R. (Doklady) Acad. Sci. URSS (N.S.) **19**, 147–149 (1938)], resp. Freudenthal [Compositio Math. **5**, 299–314 (1937)], and to Postnikov [same Doklady **71**, 1027–1028 (1950); these Rev. **11**, 676]. *H. Samelson.*

GEOMETRY

Cassina, Ugo. Ancora sui fondamenti della geometria secondo Hilbert. II. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **13(82)**, 67–84 (1949).

Cassina, Ugo. Ancora sui fondamenti della geometria secondo Hilbert. III. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **13(82)**, 85–94 (1949).

The study of the seventh edition of Hilbert's Grundlagen der Geometrie [Teubner, Leipzig, 1930] begun in an earlier note [same Rend. **12(81)**, 71–94 (1948); these Rev. **13**, 153] is here continued with an analysis and reformulation of the group of congruence axioms, and concluded with some remarks on the axiom of parallels and the axioms of continuity.

F. A. Behrend (Melbourne).

***Libois, Paul. Les espaces.** Georges Thone, Liège, 1951. 46 pp.

This small pamphlet contains a non-mathematical discussion of various types of spaces with some illustrating examples.

M. Wyman (Edmonton, Alta.).

Wyler, Oswald. Order and topology in projective planes. Amer. J. Math. **74**, 656–666 (1952).

Die topologischen Eigenschaften einer endlich-dimensionalen projektiven Geometrie über einem Körper oder Schiefkörper fliessen aus der Topologie des Koordinatenbereiches. Insbesondere ist die Gerade eine geschlossene Linie, die homöomorph ist zu dem Koordinatenbereich mit einem adjungierten Element im Unendlichen. Die Frage, ob auch die Topologie einer nicht-Desargues'schen Ebene durch die Topologie ihrer Geraden in ähnlicher Weise festgelegt ist, wird hier beantwortet für eine geordnete projektive Ebene; in ihr gelten die ebenen projektiven Verknüpfungssätze und die Axiome über das Trennen von zwei Punktepaaren einer Geraden, etwa in der Formulierung von H. S. M. Coxeter [The real projective plane, McGraw-Hill, New York, 1949; diese Rev. **10**, 729]. Dann lässt sich der Begriff des Intervalls und des Sektors erklären, und die projektive Ebene lässt sich zerlegen in drei convexe Vierecke mit gemeinsamen Ecken. Die Beziehung der Inzidenz ist eine stetige Funktion. Die Topologie der Ebene ist durch die Topologie ihrer Geraden soweit bestimmt, dass zwei geordnete projektive Ebenen mit homöomorphen Intervallen homöomorph sind. Aber die Topologie zieht selbst bei Hinzunahme das Dedekindschen Postulates noch nicht die Gültigkeit des Desargues'schen Satzes nach sich, wie das Beispiel der nicht-Desargues'schen Geometrie von Moulton zeigt.

R. Moufang (Frankfurt am Main).

Haantjes, J. Ptolemy's theorem. Simon Stevin **29**, 25–31 (1952). (Dutch)

If p_1, p_2, p_3, p_4 are any four points of euclidean three-space, the (extended) theorem of Ptolemy asserts that the three products $p_1p_2 \cdot p_3p_4$, $p_1p_3 \cdot p_2p_4$, $p_1p_4 \cdot p_2p_3$ of opposite distances satisfy the triangle inequality, and the four points are concyclic if and only if the sum of two of these products equals the third. This note contains a purely metric proof of the theorem [see also a paper by the reviewer, Revista Ci., Lima **45**, 183–193 (1943); these Rev. **6**, 12] and proofs of its analogues in spherical and hyperbolic spaces.

L. M. Blumenthal (Los Angeles, Calif.).

Seidel, J. Distance-geometric development of two-dimensional Euclidean, hyperbolical and spherical geometry. I. Simon Stevin **29**, 32–50 (1952).

This paper develops an analytic distance geometry of the euclidean plane. If a, b, c are positive constants such that the determinant

$$D(a,b,c)=\begin{vmatrix}0&1&1&1\\1&0&a&b\\1&a&0&c\\1&b&c&0\end{vmatrix}$$

is negative (that is, the numbers $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ are the sides of a proper triangle of the plane), an ordered triple (x, y, z) of non-negative real numbers is called a point if and only if the symmetric determinant obtained from $D(a, b, c)$ by annexing $1, x, y, z, 0$ as a last row and column vanishes. Distance of two points is defined as an explicit function of their coordinates (and the constants a, b, c). It is shown that the set of all such points, so metrized, satisfies a system of postulates for the euclidean plane.

L. M. Blumenthal (Los Angeles, Calif.).

Convex Domains, Extremal Problems, Integral Geometry

Straus, E. G., and Valentine, F. A. A characterization of finite dimensional convex sets. Amer. J. Math. **74**, 683–686 (1952).

It is proved that if S is a closed connected set in a Euclidean space and the smallest linear manifold containing S is n-dimensional, then S is convex if and only if for each point $x \in S$ there is a unique set $K(x)$ which is maximal relative to $[x \in K(x) \subset S$, $K(x)$ is convex and of dimension $\geqq n-1]$. First a reduction to compact S is effected, and then the argument is completed with the aid of the following lemma, due for $m=2$ to B. H. Neumann [J. London Math. Soc. **20**, 226–237 (1945); these Rev. **8**, 170] and for arbitrary m to J. W. Green and W. S. Gustin (unpublished): For each integer $m>0$ there is a constant $c_m>0$ such that if P is a hyperplane through the centroid of a bounded convex set of dimension m and volume V and V_1 is either of the volumes into which K is divided by P, then $V_1 \geqq c_m V$.

V. L. Klee, Jr. (Charlottesville, Va.).

Davenport, H. The covering of space by spheres. Rend. Circ. Mat. Palermo (2) **1**, 92–107 (1952).

Let Λ be an n-dimensional lattice of determinant $d(\Lambda)>0$, and $V(K)$ be the content of a symmetric convex body K. Let the whole of the space be covered by bodies congruent to K centered at the points of Λ, and define $\vartheta(K)$, the density of thinnest covering, to be the lower bound of $V(K)/d(\Lambda)$ for all lattices with this property. It is proved that, for the n-dimensional sphere S_n,

$$\vartheta(S_n)<\left(\frac{11\pi\epsilon}{54\sqrt{3}}+\epsilon\right)^{\frac{1}{2}n}<(1.15)^n$$

for large n, where ϵ is an arbitrary sufficiently small positive number. The method of proof is to choose a fixed integer $k>0$ and a fixed k-dimensional lattice Λ_0, to write $n=mk+h$ where $k \leqq h<2k$, and to evaluate $V(K)/d(\Lambda)$ for a particular n-dimensional lattice Λ which is defined in terms of Λ_0.

Even the simple choice $k=1$ provides a good upper limit for $\{\vartheta(S_n)\}^{1/n}$, namely $(\pi e/6)^{1/2}$, but by choosing Λ_0 to correspond to the extreme form $\sum_{i=1}^{k}\xi_i^2+2\sum_{i<j}\xi_i\xi_j$ a better upper limit is obtained which is a minimum for $k=8$, and this gives the result stated. The paper concludes with a proof of the less sharp inequality $\vartheta(S_n)<(1.621)^n$ by a method which is a modification of one used by Rogers [J. London Math. Soc. **25**, 327–331 (1950); these Rev. **13**, 323]. *R. A. Rankin* (Birmingham).

Ohmann, D. Extremalprobleme für konvexe Bereiche der euklidischen Ebene. Math. Z. **55**, 346–352 (1952).

The main result is a remarkable generalisation of a theorem of Lebesgue according to which among the domains of equal constant breadth the Reuleaux triangle has the least area. Inscribe in a closed convex curve C an affine regular hexagon $A_1\cdots A_6$. The convex domain bounded by the arcs A_1A_2, A_3A_4, A_5A_6 of C is called a general Reuleaux triangle. The author considers convex domains whose breadth in all directions equals the half breadth of C and shows that among these the general Reuleaux triangle has the least area. *L. Fejes Tóth* (Veszprém).

Sur la liaison et la division des points d'un ensemble fini. Travail collectif, rédigé par J. Łukaszewicz. Colloquium Math. **2** (1951), 282–285 (1952).

The authors give a simple and ingenious solution of the following problem: Let there be given n points in a metric space. Determine the polygonal line of least length which connects these points. *P. Erdös* (Los Angeles, Calif.).

Zalgaller, V. A. On a class of curves with curvature of bounded variation on a convex surface. Mat. Sbornik N.S. **30(72)**, 59–72 (1952). (Russian)

Let F be a non-degenerate convex surface in E^3. On F consider curves which can be decomposed into a finite number of arcs. With the notation and concepts of A. D. Aleksandov [Intrinsic geometry of convex surfaces, Moscow-Leningrad, 1948; these Rev. **10**, 619] it is shown that the following four classes of curves K coincide. 1) K has at every point well-defined right and left directions. For all decompositions of K into a finite number of simple arcs the sum of the absolute values of the total right geodesic curvatures of the open arcs and of their endpoints is uniformly bounded. 2) K has finite curvature as a curve in E^3, i.e., for all inscribed ordinary polygons the sum of the angles is uniformly bounded. 3) K can be decomposed into a finite number of arcs, each of which is the limit of a sequence of simple geodesic polygons on F for which the variation of the geodesic curvature is uniformly bounded. 4) K is the limit of geodesic polygons lying in convex surfaces tending to F, which have (on their respective surfaces) variations of the geodesic curvature which are uniformly bounded.

H. Busemann (Los Angeles, Calif.).

Algebraic Geometry

Manara, Carlo Felice. Sulla caratterizzazione delle ipersuperficie di diramazione degli S_n tripli. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **13(82)**, 140–142 (1949).

Cette note présente une conséquence d'un mémoire de Chisini et de l'auteur sur la caractérisation de la courbe de diramation des plans triples [Ann. Mat. Pura Appl. (4) **26**, 383–388 (1947); ces Rev. **10**, 322]. Il s'agit de conditions suffisantes pour qu'une hypersurface V^m_{n-1} soit variété de diramation pour un S_n triple. Cette hypersurface est supposée générique à cela près qu'elle possède comme seule sous-variété double une R^k_{n-2} de rebroussement. Si H_{n-1} désigne une section hyperplane générique, les conditions sont les suivantes:

(1) Il existe un entier $h\geqq 0$ tel que:

$$3m^2-16k=12h^2.$$

(2) Il existe une sous-variété T_{n-2} telle que:

$$R^k_{n-2}+T_{n-2}\equiv\tfrac{1}{4}(m-2h)H_{n-2}.$$

(3) Il existe une sous-variété T'_{n-2} sans composante commune avec T_{n-2} et équivalente à $T_{n-2}+hH_{n-2}$.

Dans ces conditions le S_n triple qui admet V^m_{n-1} comme variété de ramification est unique à une transformation birationnelle près. L'énoncé est seulement donné pour une surface, mais l'auteur remarque que le procédé de passage d'une courbe à une surface est en réalité un procédé récurrent sur la dimension. *L. Gauthier* (Nancy).

Godeaux, Lucien. Sur les systèmes canoniques et pluricanoniques d'une surface de genre linéaire un. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 234–243 (1952).

L'A. considère ici la surface F du 6ème ordre qui a l'équation suivante:

$$\begin{aligned}a_{11}x_1^6x_2+a_{12}x_2^6x_3+a_{12}x_3^6x_1&\\+x_4(a_{23}x_1^3x_2^2+a_{23}x_2^3x_3^2+a_{23}x_3^3x_1^2)&\\+x_4^2(a_{21}x_1^3x_2+a_{12}x_2^3x_1+a_{23}x_3^3x_2)&\\+a_1(x_1x_2x_3)^2+a_2x_4^3x_1x_2x_3+a_3x_4^6=0;&\end{aligned}$$

elle est transformée en soi par l'homographie

$$H=\begin{pmatrix}x_1 & \epsilon x_2 & \epsilon^2x_3 & \epsilon^4x_4\\ x_1 & x_2 & x_3 & x_4\end{pmatrix}$$

(où ϵ est une racine primitive d'ordre 7 de l'unité), laquelle engendre sur F une involution I_7 d'ordre 7 ayant trois points unis aux sommets O_1, O_2, O_3 du tétraèdre de référence. Après avoir construit une surface normale Φ image de l'involution I_7, l'A. détermine la structure des points de diramation O_1', O_2', O_3' qui correspondent sur Φ aux points O_1, O_2, O_3 de F, et s'en sert pour calculer les genres de la surface Φ. Il trouve que Φ contient un faisceau de courbes elliptiques $|\gamma|$; dans ce faisceau il y a une courbe particulière qui se réduit au triple $3\gamma_0$ d'une certaine courbe elliptique γ_0; celle-ci, comptée deux fois, donne la seule courbe canonique K de Φ. On a donc: $\gamma=3\gamma_0$, $|K|=2\gamma_0$, $|2K|=|\gamma_0+\gamma|$, $|3K|=|2\gamma|$, etc.; on en déduit: $p^{(1)}=1$, $p_a=p_g=1$, $P_2=2$, $P_3=3$, et, en général,

$$P_{3i}=2i+1,\quad P_{3i+1}=2i+1,\quad P_{3i+2}=2i+2.$$

E. G. Togliatti (Gênes).

Godeaux, Lucien. Sur quelques involutions rationnelles appartenant à une surface algébrique. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 426–436 (1952).

On démontre l'existence, sur une surface du septième ordre ($p_a=p_g=20$), d'une involution rationnelle cyclique ayant quatre points unis. *Author's summary.*

Godeaux, Lucien. Recherches sur les points de diramation des surfaces multiples. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **37**, 111–120 (1951).

Ce mémoire fait suite à un assez grand nombre de travaux du même auteur sur le même sujet [même Bull. (5) **35**,

15–30, 270–284, 285–292, 532–541, 636–641, 828–833, 834–840 (1949); Ann. Mat. Pura Appl. (4) **28**, 89–106 (1949); Ann. Sci. École Norm. Sup. (3) **67**, 1–13 (1950); ces Rev. **10**, 735; **11**, 205, 392, 393, 538; **12**, 737]. Considérons sur une surface algébrique S une involution cyclique I_p d'ordre premier impair p n'ayant qu'un nombre fini de points unis. Soit A l'un d'eux et A' le point de diramation correspondant de la surface Φ image de I_p. L'auteur étudie ici les conditions dans lesquelles au point A' le cône des tangentes à Φ admet au moins trois composantes. A cet effet il reprend le modèle projectif de S déjà utilisé dans les mémoires cités ci-dessus, pour lequel l'involution I_p induit dans le plan tangent à S en A une homographie périodique H:

$$x_1' = x_1, \quad x_2' = \epsilon x_2, \quad x_3' = \epsilon^\alpha x_3,$$

avec $\epsilon^p = 1$, ϵ racine primitive, α entier >1. L'étude introduit la congruence $\lambda + \alpha\mu \equiv 0 \pmod{p}$, et en faisant la division: $p = a\alpha + b$ $(b < \alpha)$ le résultat s'exprime ainsi: Il est nécessaire et suffisant qu'il existe un entier positif r tel que $(r-1)(a+b) < \alpha - 1 < r(a+b)$ et un nombre entier positif m associé, tel que

$$m(r-1)b < m\alpha < (mr-m+1)b,$$
$$(mr-m+1)(a+b) - m(\alpha-1) < p.$$

Dans ces conditions, les $m+1$ premières solutions de la congruence, rangées suivant la croissance de $\lambda+\mu$ sont:

$$\lambda = [(m+1-i)r - m + i]b - (m+1-i)\alpha,$$
$$\mu = [(m+1-i)r - m + i]a + (m+1-i), \quad i = 1, 2, \cdots, m+1.$$

Alors le cône tangent à Φ en A' comprend un cône d'ordre a, un cône d'ordre m, et en outre un ou deux autres cônes.

L'auteur termine ce mémoire par l'étude, au moyen des solutions précédentes, de la constellation des points multiples de Φ infiniment voisins de A'. *L. Gauthier.*

Godeaux, Lucien. Sur la singularité d'une surface multiple en un point de diramation. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 50–70 (1952).

L'A., qui s'est occupé beaucoup des involutions cycliques appartenant à une surface algébrique F, considère ici sur F une involution cyclique I, n'ayant qu'une nombre fini de points unis, et d'ordre $p = 9\nu^2 + 3\nu + 1$, où ν est un nombre entier positif tel que p soit un nombre premier. On suppose que F soit normale dans un espace à r dimensions, et que l'involution I soit déterminée sur F par une homographie cyclique H possédant p axes ponctuels $\sigma_0', \sigma_1', \cdots, \sigma_{p-1}'$ dont le premier seulement rencontre F (aux points unis de I). Les hyperplans passant par $\sigma_1', \cdots, \sigma_{p-1}'$ découpent sur F un système linéaire $|C_0|$ appartenant à l'involution I, et dont l'image est une surface Φ qui représente l'involution I. A un point uni A de I sur F correspond sur Φ un point de diramation A'. L'A. s'occupe ici de déterminer la singularité que la surface Φ présente au point A'. En utilisant une succession de systèmes linéaires $|C_0'|, |C_0''|, \cdots$, tous contenus dans $|C_0|$, et dont les courbes présentent au point A des multiplicités croissantes convenables, l'A. trouve, après une analyse minutieuse, que le point A' a pour Φ la multiplicité $\nu+1$, et que le cône tangent à Φ au point A' se décompose en trois parties: un cône rationnel τ d'ordre $\nu-1$ et deux plans σ_1, σ_2 qui ont chacun une droite commune avec le cône τ et qui ont en commun seulement le point A'. La surface Φ possède encore sur la droite $\sigma_2\tau$ un point double conique infiniment voisin de A'; et sur la droite $\sigma_1\tau$ un point double biplanaire infiniment voisin de A' suivi d'une succession de points doubles biplanaires qui se termine par un point double qui est biplanaire ou conique selon que ν est impair ou pair. *E. G. Togliatti* (Gênes).

Godeaux, Lucien. Sopra una particolare involuzione di Geiser. Rend. Sem. Fac. Sci. Univ. Cagliari **21** (1951), 1–3 (1952).

Turri, Tullio. A proposito dell'inesistenza di trasformazioni involutorie che lascino invariato un sistema ∞^3 di cubiche. Rend. Sem. Fac. Sci. Univ. Cagliari **21** (1951), 4–6 (1952).

S'il existe une involution plane avec point uni isolé A laissant fixe les cubiques par six points, la courbe des points unis sera d'ordre impair $2q+1 \leqq 5$; sur une droite passant par A, on a quatre points unis donc $2q+1 \leqq 4$. Turri en déduit que $4-(2q+1)$ points sur cette droite sont unis, donc que le degré de la courbe des points unis doit surpasser $2q+1$, d'où contradiction. Comme le faisait remarquer Godeaux, l'involution qu'il déterminait n'était pas générale. Ne peut-il exister une involution telle que la courbe transformée d'une droite par A, y ait en A la droite pour tangente, ce qui mettrait en défaut le raisonnement général basé sur des comptes de parité? *B. d'Orgeval* (Alger).

Turri, Tullio. Sulle trasformazioni birazionali del piano con curva di punti uniti di genere minore di 2. Rend. Sem. Fac. Sci. Univ. Cagliari **21** (1951), 11–18 (1952).

Dans une transformation birationnelle du plan, il existe au plus une courbe de points unis non rationnelle. Si cette courbe est elliptique, on peut par des transformations crémoniennes la ramener à une cubique. S'il y a des courbes de points unis rationnelles, on peut de même les réduire à un nombre fini de points. *B. d'Orgeval* (Alger).

Turri, Tullio. Sui sistemi invarianti di curve nelle involuzioni di secondo ordine. Rend. Sem. Fac. Sci. Univ. Cagliari **21** (1951), 19–26 (1952).

Détermination des systèmes invariants de courbes d'une involution de Jonquières (O^{n-1}), de Geiser, et de Bertini. Est à noter qu'à partir d'un certain ordre m (pour J, $m \geqq n$ pour G, $m \geqq 9$, pour B, $m \geqq 12$) ces courbes forment deux systèmes dont l'un formé de courbes réductibles et l'autre de courbes irréductibles dont l'A. donne les dimensions.
B. d'Orgeval (Alger).

Turri, Tullio. Sul numero dei circuiti delle curve di punti uniti nelle involuzioni piane reali del secondo ordine. Rend. Sem. Fac. Sci. Univ. Cagliari **21** (1951), 27–31 (1952).

Dans une involution plane réelle, la courbe C des points unis peut avoir des circuits dont le nombre a toutes les valeurs compatibles avec le genre, sauf 0, si l'ordre de C est impair; plus précisément on peut avoir: involution de Jonquières: d'ordre n: $1, 2, 3, \cdots, n-1$ et 0 si n impair; involution de Geiser: 0, 1, 2, 3, 4; involution de Bertini: 1, 2, 3, 4, 5. On procède, par réduction quadratique à points fondamentaux réels ou conjugués, à des types simples.
B. d'Orgeval (Alger).

Turri, Tullio. Sui punti doppi della serie caratteristica di una rete di cubiche. Rend. Sem. Fac. Sci. Univ. Cagliari **21** (1951), 32–39 (1952).

Si dans l'involution I définie par les $C_3(7A_i)$ il existe une courbe de points unis, il existe des réseaux de cubiques conservés par I. Si parmi ces réseaux, il en existe un dont les cubiques ont en un point-base une pour I une tangente commune, on ne peut à partir de ce réseau définir la quartique de diramation. Ceci remet en question la démonstration de certains résultats de Chisini [Rend. Circ. Mat. Palermo **41**, 59–93 (1916)]. *B. d'Orgeval* (Alger).

Turri, Tullio. Una proprietà delle trasformazioni antibirazionali involutorie in uno spazio qualunque. Rend. Sem. Fac. Sci. Univ. Cagliari 21 (1951), 40–42 (1952).

Par des transformations simples sur l'identité $(KM)^2=1$ l'A. établit que, dans une espace quelconque, une transformation antibirationnelle est birationnellement identique au produit d'une transformation birationnelle involutive M réelle, par la transformation K qui fait se correspondre deux points imaginaires conjugués. *B. d'Orgeval* (Alger).

Semple, J. G. On complete quadrics. II. J. London Math. Soc. 27, 280–287 (1952).

Q_L is the locus, Q_B the envelope, and Q_C the tangent complex of the general quadric in S_3; all Q_L can be mapped (without exception) by points of a linear A_9, all Q_B by points of B_9, and all Q_C by points of a manifold C_9^{92}. In part I [same J. 23, 258–267 (1948); these Rev. 10, 472] a nonsingular manifold M_9 was defined, which maps unexceptionally all complete quadrics (Q_L, Q_B, Q_C), and is in birational correspondence, though with quite different exceptional elements, with each of A_9, B_9, and C. Now the author defines a manifold M_9 (which is perhaps identical with M_9), which is the projective model of the sextic primals of A_9 which pass triply through V_3^3 (the Veronesian of quadrics in S_3) and simply through V_6^{10} (model of the unordered plane-pairs of S_3); M_9 is an unexceptional model of the complete quadrics of S_3. Further theorems on M_9, C_9^{92} and on W_9, a model of quadric locus-envelopes (Q_L, Q_B).

O. Bottema (Delft).

Semple, John G. The variety whose points represent complete collineations of S_r on S_r'. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 10, 201–208 (1951).

Dans la géométrie numérative et, plus en général, dans la géométrie algébrique, on a parfois besoin de connaître une représentation, sans aucune exception, des figures géométriques d'une classe donnée; on peut rappeler à ce sujet la représentation des éléments différentiels du premier et du second ordre du plan, ou bien celle des coniques et des quadriques complètes; dans tous ces cas on rencontre des difficultés provenant de certaines figures particulières, ou dégénérées de la classe, dont la définition même n'est pas toujours évidente. L'auteur considère ici les collinéations complètes qui lient entre eux deux espaces S_r, S'_r à r dimensions. Selon l'habitude on représente une collinéation entre les points de S_r, S'_r par un point de l'espace à r^2+2r dimensions, dont les coordonnées homogènes sont les éléments d'un déterminant $|a|$ d'ordre $r+1$. Mais si l'on passe aux collinéations complètes, c'est-à-dire, si l'on veut considérer aussi en même temps les collinéations entre les droites, les plans, etc. de S_r, S'_r, on arrive, dans le cas où le déterminant $|a|$ est nul, à des exceptions compliquées, qui conseillent la construction d'une représentation différente. On peut en obtenir une de la manière suivante: on forme, de toutes les manières possibles, un produit de r facteurs, dont le premier est un élément de $|a|$, le deuxième un mineur du deuxième ordre de $|a|$, le troisième un mineur du troisième ordre de $|a|$, etc. jusqu'à un mineur de l'ordre r de $|a|$; tous ces produits définissent dans l'espace S_{r^2+2r} un système linéaire d'hypersurfaces, qui transforme l'espace même S_{r^2+2r} en une variété qui représente sans exceptions les collinéations complètes entre S_r, S'_r. L'auteur développe rapidement les détails de cette construction et en fait l'application lorsque $r=2, 3$. Si $r=2$ le système linéaire d'hypersurfaces considéré ci-dessus se compose des formes cubiques passant par une V_4^4 de Segre de l'espace S_3; et si $r=3$ il se compose de formes du sixième degré contenant simplement une V_{11}^{20} et triplement une V_6^{20}. *E. Togliatti* (Gênes).

Roth, Leonard. Some properties of Grassmannians. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 10, 96–114 (1951).

This paper is for the most part a study of linear sections of the Grassmannian varieties $G(k, n)$ which map the spaces $[k]$ of S_n. It also includes, however, some properties of quadratic and cubic $[k]$-complexes, and of Schubert varieties $S(a_0, \cdots, a_k)$ of S_n.

In considering general linear sections V_r of dimension r of $G(k, n)$, where r, if $G(k, n)$ itself is included, takes all values from 0 to $(k+1)(n-k)$, the author finds it natural to distinguish between those V_r for which $r \geqq k(n-k)$ and those for which $r<k(n-k)$. The critical type of section $V_{k(n-k)}$ represents a linear $[k]$-congruence in S_n, such that a unique $[k]$ of the system passes through a general $[k-1]$. The section $V_{k(n-k)}$ is shown to be rational, the spaces $[k]$ of the congruence being in birational correspondence (by section) with the spaces $[k-1]$ in a fixed prime π; and it appears in fact that $V_{k(n-k)}$ projects birationally from a suitable vertex into $G(k-1, n-1)$. It is shown further, by combining the above projection with a projection of $G(k-1, n-1)$ into $S_{k(n-k)}$, that for all $r \geqq k(n-k)$ the general section V_r always projects birationally—from the same type of vertex space—into a space S_r. All these representations of V_r are of order $k(n-k)$.

For $k=1$, $r \geqq n-1$, the general V_r-section of $G(1, n)$ projects birationally into S_r from the space containing a manifold W_{n-2} on V_r, where W_{n-2} is a suitable linear section of a $G(1, n-1)$. In particular, V_{n-1} itself ($n \geqq 4$) is thus mapped on S_{n-1} by means of the system of primals $|\Phi^{n-1}|$ that pass simply through an irreducible base manifold ψ_{n-3}; the points of ψ_{n-3} are projections of lines of V_{n-1} that meet W_{n-2} ($n-1$ through each point of W_{n-2}), these lines generating a manifold R_{n-2}; and ψ_{n-3} possesses ∞^{n-3} $(n-1)$-secant lines (corresponding exceptionally to the points of W_{n-2}) that generate a fundamental primal Φ^{n-2} of $|\Phi^{n-1}|$. For $n=4$, this gives the known map of V_3^3 by cubic surfaces in S_3 passing through an ${}^9C^4$ (projection from a conic). For $n=5$, it gives a map of V_4^{14} (by projection from a Del Pezzo ${}^1F^5$) by the system of quartic primals in S_4 which pass through a rational septimic surface ${}^4F^7[4]$. This surface lies on a unique Segre 10-nodal cubic primal Φ^3. (It is also, as it happens, though this is not noted in the paper, the model of a plane curve system $C^8[2^4, 1^8]$, and it contains (a) ten plane cubics whose planes map exceptionally the ten lines of ${}^1F^5$, and (b) twenty-seven conics whose planes map quadric surfaces on V_4^{14}.)

It is shown next that the base for W_{r-1} on any general section V_r of $G(k, n)$ for which $r \geqq k(n-k)$ is a prime section. This substantial extension of Severi's well-known result for $G(k, n)$ is deduced from the exceptionally simple character of the representations of the V_r as already found. In the case of V_{n-1} ($k=1$, $r=n-1$), for example, there are, as described above, single irreducible base manifolds W_{n-2} and ψ_{n-3} on V_{n-1} and S_{n-1} respectively, and to the neighborhoods of these there correspond irreducible manifolds Φ_{n-2} and R_{n-2} on S_{n-1} and V_{n-1}. These facts imply that R_{n-2} is the complete section of V_{n-1} by a primal (of order $n-2$), and hence that a prime section is a base for $(n-2)$-folds on V_{n-1}. The result, as simple examples show, is best possible.

The next section of the paper shows, on the hypothesis that the sections concerned are regular manifolds, that (i)

the sections of $G(k, n)$ by generic primals of orders $2, \cdots, n$ have all their genera and plurigenera zero, while those by primals of order $n+1$ have genera and plurigenera unity, and (ii) the general linear sections V_{t-i} of $G(k, n)$, with $t=(k+1)(n-k)$ and $i \leqq n$, have genera zero, while V_{t-n-1} has genera unity.

There follow then some further discussions of $G(1, 5)$, $G(1, 6)$, and $G(2, 5)$. In connection with $G(1, 5)$ the author quotes Fano's observation that its general $[9]$-section V_3^{14} is birationally equivalent to a non-singular cubic primal, and he suggests a generalization. If n is odd and $|L|$ is a general linear ∞^{n-1} system of linear complexes of S_n, he observes that the ∞^{n-2} base lines of $|L|$—the system $\{n-2\}$ of lines common to n general linear complexes—are precisely the ∞^{n-2} line-centres of singular complexes in $|L|$. Hence, when complexes of $|L|$ are mapped by points of a space S_{n-1}, the primal $\Phi^{(n+1)/2}$ whose points map singular complexes of $|L|$ is a birational map of the system $\{n-2\}$, i.e., of V_{n-2}. It is suggested that $\Phi^{(n+1)/2}$ is in general free from singular points, but this appears to be in error for $n>5$ [cf. T. G. Room, The geometry of determinantal loci, Cambridge, 1938, p. 197].

On passing next to general quadratic and cubic complexes, the author has first to extend the ordinary concept of the unirationality of a manifold W_r in its coefficient field γ to that of the unirationality of W_r in a certain extension $\gamma(P)$ of γ, where $\gamma(P)$ is "defined by adjoining to γ the irrationality on which depends the determination of the generic point P of W_r". What is meant, apparently, is that if $(\xi_1, \cdots, \xi_n)$ are the coordinates of any "general" point of W_r (not belonging to a specified sub-manifold of W_r), then the coordinates of the generic point of W_r must be expressible as rational functions of r indeterminates over the field $\gamma(\xi_1, \cdots, \xi_n)$. By the use of certain criteria of unirationality which he has already published [Boll. Un. Mat. Ital. (3) 5, 330–336 (1950); these Rev. 12, 528], the author finds first that the Grassmannian model $C^2(1, n)$ of the general quadratic line-complex is unirational in $\gamma(P)$ and representable on an involution I_3, then that the model $C^2(k, n)$ of the general quadratic $[k]$-complex is representable on an I_{k+2}, and finally that the model $C^3(k, n)$ of the general cubic complex is unirational.

In the final section on Schubert varieties $S(a_0, \cdots, a_k)$, it is shown, in particular, that any such variety, as also its sections by the spaces common to $1, 2, \cdots, a_k-k$ general primes, are all birational. *J. G. Semple* (London).

Roth, Leonard. Alcune V_3 irrazionali a generi nulli. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 12, 265–269 (1952).

The author gives examples of threefolds, regular and of zero genus and plurigenera, on which the operation of adjunction terminates, which are (i) neither birational nor unirational, (ii) unirational but not birational. These threefolds possess algebraic zero divisors. The simplest example of (i) is the product of the well-known Enriques sextic surface and a curve, and of (ii) a sextic threefold in S_4 whose general prime section is an Enriques surface.

J. A. Todd (Cambridge, England).

Muracchini, Luigi. Le varietà V_3 i cui spazi tangenti ricoprono una varietà W di dimensione inferiore alla ordinaria. I. Rivista Mat. Univ. Parma 2, 435–462 (1951).

The main result of this paper consists of the determination of a certain class of V_3 whose tangent spaces generate a variety of dimension <10. These V_3 are characterised by the properties (i) that they satisfy a system of 9 or 10 linearly independent homogeneous linear differential equations of the second order; (ii) that the dual of the system of quadratics apolar to the system associated with this set of equations consists of point-cones with a common vertex. The V_3 with these properties comprise (a) cones projecting certain types of V_4, (b) developables with a directrix curve or surface, (c) varieties generated by ∞^2 S_3 tangent to a surface. Some generalisations to V_3 with $k>5$ are indicated.

J. A. Todd (Cambridge, England).

Gaeta Federico. Complementi alla teoria delle varietà algebriche V_{r-2} di residuale finito in S_r. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 12, 270–273 (1952).

The author notes the possibility of constructing a purely algebraic theory of H-ideals of finite residual and dimension $r-2$ in the ring $K[x_0, x_1, \cdots, x_r]$, K being commutative and without characteristic. He also observes that every algebraic curve S_r which is arithmetically normal in the sense of Zarski is of finite residual. *J. A. Todd.*

Sangermano, Cosimo. Sulle corrispondenze puntuali degeneri fra spazi lineari. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) 8 (1950–51), 83–89 (1952).

Let T be a point-to-point analytic correspondence among $\nu+1$ spaces, $S^1, \cdots, S^\nu, S_n$ ($\nu>1$) which may be represented by equations of the form

$$(1) \qquad y_r = f_r(x^1, \cdots, x^\nu) \quad (r=1, \cdots, n)$$

where x^λ $(\lambda=1, \cdots, \nu)$ stands for the coordinates (x_i^λ) $(i=1, \cdots, \mu_\lambda)$ in S^λ, whose dimension is μ_λ, while the y_r are coordinates of a point in an S_n. By (1), to each group of ν points, one from each space S^λ, corresponds one and only one point in S_n. Let now J_λ indicate the Jacobian matrix of the f_r's with respect to the x_i^λ's; if there is at least one J_λ whose minors of highest orders vanish for any group of $\nu+1$ corresponding points, T is said to be degenerate. The author proves the following theorem, which generalizes results found earlier for $\nu=1$ by Villa [Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 1, 228–237 (1940); 3, 209–216 (1942); same Mem. (9) 9, 19–26 (1942); these Rev. 1, 268; 8, 223; 9, 464]: A correspondence T is degenerate if and only if the $V_{2\nu\lambda}$, which is its image on Segre's $V_{n+2\nu\lambda}$—representing the groups of $\nu+1$ points, one for each space S^λ, S_n—is quasi-asymptotic $\sigma_{L,2}$ for the Segre variety. *E. Bompiani.*

Scott, D. B. Correspondences of dimensions two and three between algebraic surfaces. Proc. London Math. Soc. (3) 2, 1–21 (1952).

Les correspondances entre deux surfaces algébriques F et F^* dont l'auteur s'occupe ici peuvent être soit les correspondances algébriques S, de dimension 2, qui transforment tout point de chaque surface en un certain nombre, en général fini, de points de l'autre surface, soit les correspondances algébriques $\mathfrak{T}$, de dimension 3, qui transforment tout point de l'une des surfaces en une courbe algébrique de l'autre. Parmi les correspondances $\mathfrak{T}$ il y a des correspondances particulières que l'auteur nomme correspondances O qui jouent ici un rôle très important: elles transforment les courbes de chaque surface en courbes O, c'est-à-dire en courbes dont les Riemanniennes sont découpées par tout cycle à trois dimensions de la Riemannienne de la surface correspondante selon un cycle linéaire homologue à zéro. Après quelques généralités, l'auteur considère les corre-

spondances du type S qui sont l'intersection de deux correspondances du type $\mathfrak{T}$, $\mathfrak{T}_1$, et $\mathfrak{T}_2$. En utilisant la représentation de toutes ces correspondances sur le produit topologique $F \times F^*$, il donne en premier lieu une expression topologique de la correspondance S envisagée en fonction des cycles de base des différentes dimensions existant sur les Riemanniennes de F et de F^*. La formule ainsi obtenue permet de démontrer: que S est à valence nulle au sens de F. Severi si $\mathfrak{T}_1$ et $\mathfrak{T}_2$ le sont aussi; que la même propriété subsiste aussi lorsque $\mathfrak{T}_1$ est une correspondance O à valence nulle, tandis que $\mathfrak{T}_2$ est arbitraire; que S est à valence nulle au sens de G. Albanese si $\mathfrak{T}_1$, $\mathfrak{T}_2$ sont deux correspondances O à valence nulle. Il n'est pas facile de donner des conditions pour qu'une correspondance S soit l'intersection de deux $\mathfrak{T}$; l'auteur trouve que, si une des deux surfaces a le genre géométrique nul, alors toute S équivaut topologiquement à une somme, avec des coefficients qui sont des nombres entiers ou fractionnaires, d'un nombre fini de correspondances-intersections. Dans le cas général, on peut associer à une S quelconque un nombre entier h et une S_0 de valence nulle au sens de Albanese de sorte que $hS+S_0$ équivaille à une intersection de deux $\mathfrak{T}$. Le cas où les deux surfaces F, F^* coïncident donne lieu à des questions particulières. Chaque $\mathfrak{T}$ donne lieu dans ce cas à une correspondance caractéristique, qui est l'intersection $\mathfrak{T} \wedge \mathfrak{T}^{-1}$ de $\mathfrak{T}$ avec $\mathfrak{T}^{-1}$; l'auteur en donne l'expression topologique, ainsi que l'expression topologique de la courbe U des points unis d'une $\mathfrak{T}$. En utilisant ces formules et en appliquant la définition de valence (non nulle) que l'auteur même a donnée dans un mémoire précédent [Proc. Cambridge Philos. Soc. **45**, 342–353 (1949); ces Rev. **10**, 733], on trouve que si $\mathfrak{T}$ est à valence, alors $\mathfrak{T} \wedge \mathfrak{T}^{-1}$ l'est aussi, au sens de Albanese. La même propriété a lieu pour deux $\mathfrak{T}$ quelconques, ou bien pour une $\mathfrak{T}$ quelconque et pour une $\mathfrak{T}$ du type O; mais dans ces deux derniers cas les valences, dans les deux sens, de l'intersection ne sont pas en général égales. On revient ainsi à une question que l'auteur a déjà considérée [Pont. Acad. Sci. Acta **14**, 57–61 (1951); ces Rev. **13**, 272], c'est-à-dire la validité, ou non, du théorème général de Albanese selon lequel les valences de deux correspondances S et S^{-1} sont toujours égales. En appliquant les résultats obtenus, l'auteur donne un exemple, sur une surface réglée, d'une correspondance S qui ne vérifie pas le théorème de Albanese; mais cet exemple présente des caractères très particuliers; l'auteur pense que l'on peut donner des exemples moins particuliers.

E. G. Togliatti (Gênes).

Severi, Francesco. Una nuova visione della geometria sopra una curva. Pont. Acad. Sci. Acta **14**, 143–152 (1951). (Italian. Latin summary)

The author develops the birational geometry on an algebraic curve without presupposing a previous reduction of the singularities and gives in the present note a method which leads a posteriori to a global reduction of the singularities. (Author's summary.)

O. Zariski.

Severi, Francesco. Ulteriori complementi alla teoria della base. Rend. Circ. Mat. Palermo (2) **1**, 71–87 (1952).

The first part of this paper is concerned, in the main, with obtaining a criterion for algebraic equivalence of two curves on a surface in a birationally invariant form, namely: necessary and sufficient conditions that two virtual curves A, B on a surface are pseudo-equivalent are: (i) there exists a (possibly virtual) curve C of positive degree such that the virtual intersection numbers $[AC]$, $[BC]$ are equal; (ii) $A-B$ has zero virtual degree. This very general formulation includes all previous results of similar type.

The latter part of the paper is concerned with subgroups of the group of algebraic equivalence on a surface. The author distinguishes three types. Any such subgroup is generated by a finite number of algebraically independent curves $C_1, \cdots, C_l$. The subgroups of the first type are those which contain at least one curve of positive degree. Writing $a_{ij}=[C_iC_j]$ the author shows that, for subgroups of this type, the quadratic form $\phi=\sum a_{ij}x_ix_j$ is non-singular, and that its index of inertia is 1 (i.e., that the equivalent diagonal form has one positive term). The subgroups of the second type are those in which every element (save the zero) is of negative degree; in this case ϕ is non-singular and negative definite. The subgroups of the third type are those without elements of positive degree, but containing elements of zero degree which are not zero-divisors. In this case ϕ is singular, of rank $l-h$ $(1 \leqq h \leqq l)$ and negative semi-definite. [Reviewer's remark. The reviewer has shown that in fact the only possibility is $h=1$. See J. London Math. Soc. **26**, 73–74 (1951); these Rev. **12**, 529.] The paper concludes with an example which shows that we can have, on a surface, two curves A, B of the same positive order with $[A^2]=[AB]=[B^2]=0$ without, for any λ, λA and λB belonging to the same irreducible totality of linear systems (though A and B are certainly algebraically equivalent).

J. A. Todd (Cambridge, England).

Rosenlicht, Maxwell. Equivalence relations on algebraic curves. Ann. of Math. (2) **56**, 169–191 (1952).

Study of the following equivalence relation (stronger than linear equivalence) between divisors of a field K of algebraic functions of one variable over k: two divisors A and B are equivalent if $A-B$ is the divisor of a function f which is a unit in a given semi-local subring $\mathfrak{o}$ of K. The paper begins with an algebraic study of the semi-local subrings of K, and of their relations with valuations of K. Then, given a semi-local subring $\mathfrak{o}$ of K, $\mathfrak{o}$-repartitions, and $\mathfrak{o}$-differentials are defined as in the classical theory (but the valuations of K whose rings contain $\mathfrak{o}$ are neglected), and a Riemann-Roch type theorem is proved where the genus g is replaced by the $\mathfrak{o}$-genus $\pi=g+\delta$ ($\delta=\dim_k \bar{\mathfrak{o}}/\mathfrak{o}$, $\bar{\mathfrak{o}}$ being the integral closure of $\mathfrak{o}$). If C is the smallest positive divisor such that $v(C) \geqq -v(\omega)$ for every $\mathfrak{o}$-differential ω and every valuation v of $\mathfrak{o}$, then the set $\mathfrak{c}$ of the functions f such that $v(f) \geqq v(C)$ for every valuation v of $\mathfrak{o}$ is the conductor of $\mathfrak{o}$ in $\bar{\mathfrak{o}}$; the degree $d(C)$ satisfies $d(C) \leqq 2\delta$ (the equality holding when the $\mathfrak{o}$-differentials form an $\mathfrak{o}$-module of rank one; then $\dim_k \mathfrak{o}/\mathfrak{c}=\dim_k \bar{\mathfrak{o}}/\mathfrak{o}$). Cases where $d(C)=2\delta$ are studied: $\mathfrak{o}$ is the local ring of a point on a curve which is a complete intersection (a plane curve for example), K is quasihyperelliptic (i.e., there exists $x \neq 0$ such that $[K: k(x)]=2$). This last notion enters also into play when discussing the curve defined by the ratios of the $\mathfrak{o}$-differentials of the first kind. The behaviour under ground field extensions of the above defined notions is studied. An extension of the theory to reducible curves is sketched, the field K being replaced by a direct sum of algebraic function fields, and the genus g by $g_1+\cdots+g_s-h+1$.

P. Samuel (Ithaca, N. Y.).

Zariski, Oscar. Complete linear systems on normal varieties and a generalization of a lemma of Enriques-Severi. Ann. of Math. (2) **55**, 552–592 (1952).

The first of the three parts constituting the present paper deals with the theory of complete linear systems on absolutely normal varieties (i.e., on varieties which are normal

with respect to an algebraically closed field of definition). Some of the material thus obtained extends results well known in the classical case; but the formulation and the proofs in the abstract case are not always obvious.

Part II establishes the following remarkable theorem. Let V be an irreducible absolutely normal variety and let C_m denote the intersection of V with a general hypersurface of order m (general with respect to some given algebraically closed field of definition of V). If $|D|$ is an arbitrary complete linear system on V, there exists an integer $m(D)$ (depending on D) such that the system $|D|$ cuts out a complete linear system on C_m if $m \geqq m(D)$. This theorem, in the case when V is a surface over the complex field and $|D|$ is the adjoint system of C_i (i being an arbitrary integer), was previously obtained by Severi as a key lemma for proving the Riemann-Roch theorem upon an algebraic surface [cf. Severi, Atti Accad. Sci. Torino **40**, 766–776 (1905), §3]; Severi has also outlined a proof of his lemma for the case of an arbitrary variety, of dimension r, with ordinary singularities in S_{r+1} [cf. Severi, Rend. Circ. Mat. Palermo **28**, 33–87 (1909)]. The present extension—concerning normal varieties—follows a quite different path, having points of contact with an attempt of extension made by Enriques in proving the Riemann-Roch theorem [Enriques, Le superficie algebriche, Zanichelli, Bologna, 1949, pp. 128–134; these Rev. **11**, 202]. Here it is shown first that it suffices to prove the theorem above in the special case when $|D|$ coincides with one of the complete linear systems $|C_i|$ ($i=1, 2, \cdots$), and then the result is reached in this case by means of some ingenious ideas.

Part III develops the concept of the virtual arithmetic genus p_a of an arbitrary $(r-1)$-cycle on an r-dimensional absolutely irreducible and absolutely normal variety V, by the intermediary of the Hilbert characteristic function [concerning this, cf. also the recent paper (not quoted here) by Severi, Ann. Mat. Pura Appl. (4) **32**, 1–81 (1951); these Rev. **13**, 979]. As a consequence, it is proved that, if D is any $(r-1)$-cycle on V, then, for all sufficiently large values of m, the complete linear system $|D+C_m|$ (where C_m is the intersection of V with a form of order m) has the dimension

$$(-1)^r\{p_a(V)+p_a(-D-C_m)\}-1.$$

Next the canonical system $|K|$ and the adjoint systems are introduced on V by means of the r-fold differentials of the first kind (without considering, however, the possibility that no such differentials exist), and the well known and important question of proving Severi's equality $p_a(V)=P_a(V)$ is discussed, $P_a(V)$ being here defined by

$$P_a(V)=(-1)^r\{p_a(V)+p_a(-K)-1\}$$

[cf. also §§25–26 of the paper by Severi last quoted; it seems that a rigorous (as yet unpublished) proof of that equality, in the classical case, has been meanwhile obtained by K. Kodaira]. Finally, the Riemann-Roch theorem is obtained for normal surfaces. The proof of this theorem is thus made independent of the assumption that the surface under consideration has no singularity; and this is of special importance in the case of non-zero characteristic, when the existence of a non-singular model is still an open question. [Cf. B. Segre, Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8° **14**, no. 8 (1936), §55]. *B. Segre* (Rome).

Matsusaka, Teruhisa. On the algebraic construction of the Picard variety. Proc. Japan Acad. **28**, 5–8 (1952).

A sketch is given of an algebraic construction of the Picard and Albanese varieties attached to a normal variety V over a field of arbitrary characteristic (for a non-singular variety over the complex field, see Igusa [Amer. J. Math. **74**, 1–22 (1952); these Rev. **13**, 680]; the notations and definitions used here are the same as in this review). Let S be an irreducible algebraic system of positive divisors on V, let C be a generic linear section of dimension 1 of V, let J be the jacobian variety of C, and f the canonical mapping of C into J; the formula $h(X)=S(f(X\cdot C))$ $(X \in S)$ defines a rational mapping of S into J; S can be chosen such that the image $h(S)$ is an abelian subvariety B of J containing (modulo translations) every image $h'(S')$ of the same type (the reviewer guesses it is sufficient to take an S such that dim $(h(S))$ is maximal). Every divisor $D \in G_a(V)$ is then linearly equivalent to a divisor of the form $X'-X''$ $(X', X'' \in S)$. The Picard variety P of V is defined as the variety having the complete linear systems contained in S as points; it is isogeneous to B; there is a field of definition k of P and V such that, if $D \in G_a(V)$ is rational over $K \supset k$, then the corresponding point of P is rational over K; the converse is not stated, although it is essential for proving the uniqueness of the Picard variety in characteristic $p \neq 0$. The variety B is also used for constructing the Albanese variety A. The abelian varieties P and A are birational invariants of V, and may be imbedded in projective spaces. The proofs are either sketchy or absent. *P. Samuel* (Ithaca, N. Y.).

Igusa, Jun-ichi. Some remarks on the theory of Picard varieties. J. Math. Soc. Japan **3**, 345–348 (1951).

In a previous paper [Amer. J. Math. **74**, 1–22 (1952); these Rev. **13**, 680], the author attached to every non-singular projective variety V over the complex field two abelian varieties A^q and P^q. Here is proved the existence of an isomorphism from the divisor-class group $G_a(V)/G_l(V)$ onto P, and of a field of definition K of V and P such that, if $X \in G_a(V)$ is rational over a field L containing K, then its image in P is rational over L; this fact implies the uniqueness of the Picard variety P (since the characteristic of the complex field is 0). Furthermore the Albanese variety A is the "universal" variety for mappings of V into abelian varieties: there exists an application f_0 from V to A whose image generates A, and, for every application f of V into an abelian variety B, there exists a homomorphism h of A into B such that $f=hof_0+\text{const}$. From these facts are derived natural isomorphisms between the module $C(V, V')$ of correspondence classes between two varieties V and V' and the modules $H(A, P')$, $H(A', P)$ of homomorphisms from A to P' and A' to P (A, P and A', P' being the abelian varieties attached to V and V'). *P. Samuel* (Ithaca, N. Y.).

Igusa, Jun-ichi. On a theorem of Lueroth. Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **26**, 251–253 (1951).

Three proofs are given of the following generalization of Lüroth's theorem: every one-dimensional subextension K of a purely transcendental extension $k(x_1, \cdots, x_n)$ is a simple transcendental extension of k. Let C be a model of K. Since it is easily proved that C has a rational point over k, it is sufficient to prove that its genus is 0. The first proof of this fact considers the differential forms of the first kind on C. The second one uses the canonical mapping of C into its jacobian variety. The third one notices that the mapping of L^n onto C defined by $f(x_1, \cdots, x_n)=P$ (P a generic point of C such that $K=k(P)$) has valence zero. [Reviewer's remark: This generalization may also be deduced from Lüroth's theorem by simple and purely field-theoretic considerations.] *P. Samuel* (Ithaca, N. Y.).

Kodaira, Kunihiko. Arithmetic genera of algebraic varieties. Proc. Nat. Acad. Sci. U. S. A. **38**, 527–533 (1952).

Let V_n be an irreducible non-singular algebraic variety of dimension n imbedded in a projective space. Let D be an arbitrary divisor on V_n, and let $\{E\}$ be an ample linear system on V_n (that is, a system consisting of all hyperplane sections of V_n in some ambient projective space). Then there exists a polynomial $v(h; D, V_n)$ in h of degree n such that $\dim|D+hE|_{V_n}=v(h; D, V_n)$ for large integers h. The arithmetic genera $p_a(V_n)$, $P_a(V_n)$ of V_n are defined by the formulas:

$$p_a(V_n)=(-1)^n v(0; 0, V_n),\quad P_a(V_n)=v(0; K, V_n)+1-(-1)^n.$$

Here K denotes a canonical divisor on V_n. Now let $g_k(V_n)$ $(1\leqq k\leqq n)$ denote the number of linearly independent k-ple differentials of the first kind attached to V_n, and set $a(V_n)=\sum_{k=1}^{n}(-1)^{n-k}g_k(V_n)$. Severi [Rend. Circ. Mat. Palermo **28**, 33–87 (1909), p. 87] conjectured that $P_a(V_n)=a(V_n)$, and the author proves this conjecture.

Let $f(V_n)$ be a numerical-valued functional defined for all irreducible non-singular algebraic varieties V_n of arbitrary dimension n. We say that f is an a-functional if the following four conditions are satisfied: (i) For every non-singular curve V_1, $f(V_1)=0$; (ii) for every projective space S_n, $f(S_n)=0$; (iii) if S and T are non-singular prime divisors on V_n and if $|S|_{V_n}=|T|_{V_n}$ on V_n, $n\geqq 2$, then $f(S)=f(T)$; (iv) let U, S, T be non-singular prime divisors on V_n; if $|U|_{V_n}=|S+T|_{V_n}$ on V_n, $n\geqq 3$, and if S cuts out on T a non-singular prime divisor $S\cdot T$, then $f(U)=f(S)+f(T)+f(S\cdot T)$. The author proves that $a(V_n)$ is an a-functional.

Finally, let S_d be a d-dimensional projective space, and let $H(l_1)$, $H(l_2)$, $\cdots$, $H(l_{d-n})$, $\cdots$ be non-singular hyperspaces in S_d of respective orders $l_1, l_2, \cdots, l_{d-n}, \cdots$ such that each $H(l_{d-n})$, $n\leqq d-1$, cuts out on the intersection $C_{n+1}=H(l_1)\cap H(l_2)\cap\cdots\cap H(l_{d-n-1})$ a non-singular prime divisor $C_n=H(l_{d-n})\cap C_{n+1}$. Each C_n, $1\leqq n\leqq d$, is an irreducible non-singular variety of dimension n, which is called a complete intersection of dimension n. A variety V_n which can be imbedded without singularities in a complete intersection C_{n+1} will be called a C-variety, and it is proved that $a(V_n)=p_a(V_n)$ for every C-variety. The author states on p. 532, lines 14–16, that an arbitrary irreducible, non-singular variety V_n is a C-variety (from which it follows that $a(V_n)=p_a(V_n)$ for every irreducible non-singular V_n), but he has pointed out to the reviewer that this statement cannot be justified. *D. C. Spencer* (Princeton, N. J.).

Kodaira, Kunihiko. On the theorem of Riemann-Roch for adjoint systems on Kählerian varieties. Proc. Nat. Acad. Sci. U. S. A. **38**, 522–527 (1952).

Let $\mathfrak{M}$ be a compact Kähler manifold of (complex) dimension m. Suppose given a (possibly reducible) compact analytic subvariety S of $\mathfrak{M}$ of dimension $m-1$ and denote by $\mathfrak{W}(S;\mathfrak{M})$ the linear space consisting of all mermorphic m-ple differentials W on $\mathfrak{M}$ which are multiples of $-S$ in the sense that $(W)+S\geqq 0$, (W) being the divisor of W. The problem of Riemann-Roch for the adjoint system of S consists in expressing the dimension of the linear space $\mathfrak{W}(S;\mathfrak{M})$ in terms of numerical characteristics of $\mathfrak{M}$ and S. If there exists at least one non-trivial meromorphic m-ple differential W_0 on $\mathfrak{M}$, the canonical divisor K on $\mathfrak{M}$ may be defined by $K=(W_0)$. Then the dimension of $\mathfrak{W}(S;\mathfrak{M})$ is equal to the dimension of the adjoint system $|K+S|$ of S increased by 1. In this note the problem is solved in the following two cases: (a) S is an irreducible non-singular variety; (b) S consists of two irreducible components S_1, S_2 which are both non-singular and the intersection $\Gamma=S_1\cap S_2$ is an irreducible non-singular variety having the intersection multiplicity 1 in the sense that at each point of Γ the two varieties S_1 and S_2 have distinct tangent planes.

Let $\mathfrak{G}_\nu(\mathfrak{M})$ be the linear space of all ν-ple differentials of the first kind on $\mathfrak{M}$, and let $g_\nu(\mathfrak{M})$ be the dimension of $\mathfrak{G}_\nu(\mathfrak{M})$. In case (a) it is shown that

$$\dim\mathfrak{W}(S;\mathfrak{M})=g_m(\mathfrak{M})-g_{m-1}(\mathfrak{M})+g_{m-1}(S)+l$$

where l is the dimension of the subspace of $\mathfrak{G}_{m-1}(\mathfrak{M})$ consisting of those differentials which vanish on S. In case (b) it is shown that

$$\dim\mathfrak{W}(S;\mathfrak{M})=g_m(\mathfrak{M})-g_{m-1}(\mathfrak{M})+g_{m-2}(S)+\sum_{\lambda=1}^{2}\{g_{m-1}(S_\lambda)-g_{m-2}(S_\lambda)\}+g_{m-2}(\Gamma)+l$$

where l has the same meaning as above and where $g_{m-2}(S)$ is the dimension of the linear space

$$\{\alpha_1+\alpha_2\,|\,\alpha_\lambda\in\mathfrak{G}_{m-1}(S_\lambda)\quad(\lambda=1,2),\ \alpha_{1\Gamma}=\alpha_{2\Gamma}\}.$$

Here $\alpha_{\lambda\Gamma}$ denotes the restriction of α_λ to Γ.

The method is that of harmonic integrals and moments, and is a generalization to dimension m of the method described in section 6 of the author's recent paper on Kähler surfaces [Amer. J. Math. **73**, 813–875 (1951); these Rev. **13**, 981]. Let $\tilde{S}$ be a non-singular model of S which coincides with S in case (a), and in case (b) consists of two irreducible components $\tilde{S}_1$ and $\tilde{S}_2$ which are mapped biregularly onto S_1 and S_2 respectively. Let $\mathfrak{p}\in\mathfrak{M}$, $p\in\tilde{S}$, $\mathfrak{p}=\phi(p)$, and let $z=(z^1,\cdots,z^m)$ be local coordinates on $\mathfrak{M}$ with center $\mathfrak{p}$, $u=(u^1,\cdots,u^{m-1})$ local coordinates on $\tilde{S}$ with center p. Then the mapping $\tilde{S}\to S$ has the local representation $z^\alpha=\phi_p{}^\alpha(u^1,\cdots,u^{m-1})$. The subvariety S is defined at each point $\mathfrak{p}$ by a minimal local equation $R_\mathfrak{p}(z)=0$, and it is readily verified that the expression

$$\sigma_p=(-1)^{\alpha-1}d\phi_p{}^1\cdots d\phi_p{}^{\alpha-1}d\phi_p{}^{\alpha+1}\cdots d\phi^m/\partial_\alpha R_{\phi(p)}(\phi_p(u)),$$

$\partial_\alpha=\partial/\partial z^\alpha=\partial/\partial\phi_p{}^\alpha$, is independent of the choice of α $(\alpha=1,2,\cdots,m)$. If $W=W_{12\cdots m}dz^1\cdots dz^m$ is an m-ple differential on $\mathfrak{M}$, $(W)+S\geqq 0$, the expression

$$w=\Re(W)=(R_{\phi(p)}W_{12\cdots m})(\phi_p(u))\cdot\sigma_p$$

is the Poincaré residue of W on its polar variety S. As currents on $\mathfrak{M}$, W and $w=\Re(W)$ are of types $(m,0)$ and $(m,1)$ respectively and they are related by the formula $dW=2\pi i\Re(W)$. Let Σ (the conductor) be the divisor on $\tilde{S}$ which in a neighborhood $U(p)$ of $p\in\tilde{S}$ is defined by the relation $\Sigma=-(\sigma_p)$, and let $\mathfrak{w}(\Sigma)$ be the linear space of $(m-1)$-ple differentials on $\tilde{S}$ which satisfy $(w)+\Sigma\geqq 0$. In case (a), $\Sigma=0$, while in case (b), $\Sigma=\tilde{\Gamma}=\phi^{-1}(\Gamma)$. As a differential on $\tilde{S}$, $w=\Re(W)\in\mathfrak{w}(\Sigma)$, while, as a current on $\mathfrak{M}$, $dw=Hw=0$. On the other hand, given any $w\in\mathfrak{w}(\Sigma)$ satisfying $dw=Hw=0$, let Λ be the Hodge-Weil operator mapping currents of type (ρ,σ) into currents of type $(\rho-1,\sigma-1)$ and let G be the Green's operator of de Rham on $\mathfrak{M}$. Then $W=2\pi(d\Lambda+i\delta)Gw\in\mathfrak{W}(S;\mathfrak{M})$ with $dW=2\pi i w$. The above Riemann-Roch formulas are obtained by evaluating the number of independent conditions imposed on an element $w\in\mathfrak{w}(\Sigma)$ in order that $dw=Hw=0$.

D. C. Spencer (Princeton, N. J.).

Differential Geometry

van der Waag, E. J. Sur les courbures. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 92–103, 104–110 (1952).

Ces deux notes sur les courbures constituent une continuation des publications antérieures suivantes: C. R. Acad. Sci. Paris **231**, 1026–1027, 1120–1122 (1950); mêmes Proc. **54**, 390–403, 404–417 (1951); **55**, 41–51, 52–62 (1952); ces Rev. **12**, 633; **13**, 771, 867. L'auteur poursuit l'étude des fondements de la Géométrie Infinitésimale Directe abordée par le rapporteur dans la seconde partie de sa Thèse [La méthode métrique en calcul des variations, Hermann, Paris, 1941; ces Rev. **7**, 67], dont il étend, approfondit et rectifie les résultats. Par définition, une courbe (orientée) euclidienne continue et rectifiable k possédant partout dans un voisinage de O une tangente ordinaire, admet en O une courbure "classique" $=\lambda$, lorsque le rapport $\Delta\alpha/\Delta s$ entre l'angle $\Delta\alpha$ des tangentes orientées en O et en P, et la longueur Δs de l'arc OP a une limite $=\lambda$ quand P tend vers O. Si, dans le cas du plan, $\Delta\alpha$ représente l'angle orienté, Δs la mesure algébrique de l'arc OP, la courbure ainsi définie est dite "signée". Il y a parallèlisme entre la courbure classique signée et la dérivée seconde classique; la définition au moyen du point caractéristique de la normale est applicable. L'exemple de la courbe $z=x^3$ pour $x\geqq 0$, $z=-x^2$ pour $x\leqq 0$ fait aussitôt comprendre le caractère pathologique de la courbure dite classique (non signée). Un critère d'existence de cette courbure est donné quand k est représentée par $y=f(x)$, $z=g(x)$ dans un voisinage de O, l'axe des x étant la tangente en O. Un ensemble est dit posséder au point d'accumulation O une courbure de Alt $=\kappa_0$ si la courbure $\kappa(P, O, Q)$ du triplet (P, O, Q) tend vers κ_0 quand P et Q tendent sur E vers O. Un critère analytique d'existence de κ_0 est donné quand E a une tangente en O prise comme axe des x. L'interprétation erronée par le rapporteur de son exemple $y=x^3\sin(1/x)$, $z=x^3\cos(1/x)$ si $x\neq 0$, $y=z=0$ si $x=0$, est rectifiée ainsi: A l'origine O la courbure de Alt n'existe pas, mais la courbure classique existe et $=1$. (Erratum: Changer 1 en 4, p. 100, ligne 8.) Le rapporteur avait prétendu qu'une condition nécessaire et suffisante pour qu'une courbe rectifiable k possédant une tangente ordinaire en O admette en O une courbure de Alt $=\kappa_0$, était que le rapport $\Delta\alpha/\Delta s$, considéré aux points P où existe la tangente ordinaire, ait une limite $=\kappa_0$ lorsque $P\to O$. L'auteur construit l'exemple d'une courbe gauche k admettant en O une courbure de Alt, pour laquelle le rapport $\Delta\alpha/\Delta s$ n'a pas de limite quand $P\to O$. Il montre en outre que l'existence de la courbure κ_0 et du plan osculateur de Alt en O impliquent $\lim \Delta\alpha/\Delta s=\kappa_0$. [Remarque du rapporteur: La condition incriminée devient correcte dans le cas du plan si l'on envisage $\Delta\alpha$ et Δs algébriquement. Il serait intéressant de savoir s'il en est ainsi dans le cas général, l'existence du plan osculateur de Alt π_0 étant postulée, si l'orientation de l'angle des tangentes est définie vis-à-vis de π_0 supposé orienté.] L'existence d'une courbure de Alt infinie pour une courbe plane k: $x=x(t)$, $y=y(t)$ en un point O est étudiée analytiquement. Si k admet en O une courbure de Alt infinie, un voisinage de O sur k est une courbe rectifiable, pourvu qu'un voisinage de O soit dépourvu de points multiples. La courbe k: $y=t\sqrt{t}$, $x=t$ si $t\geqq 0$, $y=-t$, $x=-t\sqrt{-t}$ si $t<0$, possède en O une courbure de Alt infinie, les demi-tangentes en O à droite et à gauche étant respectivement OY et OX.

C. Pauc (Nantes).

van der Waag, E. J. Sur les courbures. III, IV, V. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 275–286, 287–295, 296–303 (1952).

Ces trois notes représentent la conclusion des travaux de l'auteur [voir l'analyse ci-dessus] constituant sa Thèse [Analyse comparée des notions fondamentales de la géométrie différentielle des courbes, Amsterdam, 1952]. Une courbe (orientée) k de l'espace euclidien E_3 est dite posséder en O une courbure de Gödel $=\kappa_0$ lorsque $\lim \kappa(P', O, P'')=\kappa_0$ si sur la courbe $P'<O<P''$ et si les points P', O, et P'' de l'espace E_3 sont distincts. [Remarque du rapporteur: Une définition précise exige des notations différentes pour les "points sur k" et leurs supports.] Si k admet en O (intérieur à k) une courbure de Gödel, elle admet en O une tangente; cette tangente est ordinaire s'il existe un voisinage $\widehat{Q'Q''}$ de O sur k tel que les arcs (segments) $\widehat{Q'O}$ et $\widehat{OQ''}$ n'aient que le point O de E_3 en commun. L'exemple de la courbe $x=t$, $y=t^2$ si $t\geqq 0$, $x=-t$, $y=t^2$ si $t<0$, montre que la condition supplémentaire est nécessaire. [Remarque du rapporteur. Cette condition peut être remplacée par la suivante: Il n'existe aucun voisinage $\widehat{Q'Q''}$ de O sur k tel que les segments $\widehat{Q'O}$ et $\widehat{OQ''}$ de k aient même support ponctuel.] Pour un ensemble E de E_3 admettant en O une tangente ordinaire, la courbure de Gödel est définie comme la limite quand elle existe, de la courbure $\kappa(P', O, P'')$ quand P' et P'' convergent vers O sur E de part et d'autre de O relativement à la tangente. Les deux notions de courbure de Gödel sont comparées. Si k admet en O une tangente ordinaire, l'existence du cercle osculateur tangent équivaut à l'existence du cercle osculateur de Gödel pourvu que les rayons des cercles soient finis et $\neq 0$. Un exemple montre que k peut posséder en O une courbure de Alt(finie) sans que le plan passant par la tangente en O et un point voisin sur k ait une limite(plan osculateur I). Si un continu C, localement connexe en O, possède en O deux plans osculateurs I à droite et à gauche et une courbure de Gödel $\neq 0$(ce qui implique que la tangente en O soit ordinaire), C admet en O un plan osculateur de Gödel. Si un ensemble E admet en O une courbure de Gödel (finie) κ_0 et un plan osculateur de Gödel π, pour toute suite régulière de triplets (P_1^n, O, P_2^n) de points de E convergeant vers O, $\lim \kappa(P_1^n, O, P_2^n)=\kappa_0$ et $\lim \text{plan}(P_1^n, O, P_2^n)=\pi$. Si le continu C admet en O une courbure de Gödel(finie) $=\kappa_0$, pour toute suite régulière de triplets (P_{1n}, O, P_{2n}) convergeant vers O, $\lim \kappa(P_{1n}, O, P_{2n})=\kappa_0$; un exemple montre que ce résultat cesse d'être vrai si C est un ensemble quelconque. Si un continu C est tel que, pour toute suite régulière $(P_n', O, P_n'')\to 0$, $\lim \kappa(P_n', O, P_n'')=\kappa_0$ (fini), la tangente t à C en O existe, la courbure tangente en O existe et $=\kappa_0$; dans le cas où t est ordinaire, la courbure de Gödel existe et $=\kappa_0$. La courbe

$$\begin{aligned} y&=x^3\cos(-\ln x)^{1/2} && y=0 \\ &\text{si } 0<x\leqq 1, && \text{si } x=0, \\ z&=x^3\sin(-\ln x)^{1/2} && z=0 \\ y&=x^3\cos(-\ln(-x))^{1/2} \\ &\text{si } -1\leqq x<0 \\ z&=x^3\sin(-\ln(-x))^{1/2} \end{aligned}$$

admet à l'origine une courbure de Menger tandis que le plan osculateur de Menger n'existe pas. Pour une fonction continue $y=f(x)$ définie sur un intervalle $(-\epsilon, +\epsilon)$ l'existence de la limite (finie) de la différence divisée $[x_1, x_2, x_3; f]$ quand les triplets (x_1, x_2, x_3) convergent régulièrement vers 0, implique l'existence de la limite quand x_1, x_2, x_3 convergent vers 0, assujettis seulement à être distincts. Toutes les démonstrations sont données explicitement. *C. Pauc.*

Bouligand, Georges. Sur les parallélismes généralisés. J. Math. Pures Appl. (9) **31**, 141–159 (1952).

L'espace considéré est l'espace vectoriel à $n+1$ dimensions $V=V_{n+1}$. $x_1, x_2, \cdots, x_n, z$: composantes vis-à-vis d'une base fixée du vecteur $\overrightarrow{Om}$. TC: transformation de contact. TCP: TC conservant le parallélisme des éléments de contact. w: direction de plan. Dans une transformation $\overrightarrow{OM}=\Theta(m, w)$, le plan tangent en M à l'image d'une surface S: $z=z(x_1, \cdots, x_n)$ est en général défini par les vecteurs

$$\vec{V}_i=\frac{\partial M}{\partial x_i}+p_i\frac{\partial M}{\partial z}+p_{ii}\frac{\partial M}{\partial p_1}+\cdots+p_{in}\frac{\partial M}{\partial p_n},$$

qui font intervenir l'élément de contact du second ordre (m, p_i, p_{ij}) de S en m; cet élément est dit "critique" si les vecteurs $\vec{V}_i$ ne sont pas linéairement indépendants. Utilisant la polarité vis-à-vis d'un paraboloide d'axe parallèle à l'axe des z, l'auteur obtient les équations d'une TCP quelconque et les applique aux TC définies par $\overrightarrow{OM}=\overrightarrow{Om}+l\vec{\tau}(m, p)$, l représentant une constante. Dans le cas où τ ne dépend pas de m, la TCP correspondante peut être définie comme une transformation par surfaces parallèles dans l'espace de Minkowski $(V, \mathfrak{T})$ d'indicatrice $\mathfrak{T}$ à plan tangent continu mais non nécessairement symétrique ou convexe. Puis $\mathfrak{T}$ est supposée convexe, enveloppe des plans $z=p_1x_1+\cdots+p_nx_n-f$, où f est une fonction continue, à dérivées secondes continues en $p_1, \cdots, p_n$. A tout l correspond un parallélisme minkowskien. L'image S_l de S s'obtient en portant sur la demi-normale minkowskienne intérieure ou extérieure suivant le signe de l, un segment de $\mathfrak{T}$-mesure algébrique $=l$. Sur chaque génératrice de la congruence des $\mathfrak{T}$-normales (intérieures ou extérieures) de S, il y a n points focaux correspondant aux valeurs de l pour lesquelles (m, p_i, p_{ij}) est critique. [Remarque du rapporteur: Ce théorème découle de la théorie des points conjugués de Kneser puisque les $\mathfrak{T}$-normales forment une famille de géodésiques transversales à S, et se relie à la version du théorème de Meusnier et de l'indicatrice de Dupin pour une surface dans un espace de Minkowski, donnée par H. Rund dans sa Thèse, Cape Town, 1950.] Des indications sont données concernant l'extension de la théorie du parallélisme minkowskien au cas où $\mathfrak{T}$ est, sans plus, la frontière d'un corps convexe contenant l'origine.

C. Pauc (Nantes).

Pöschl, Theodor. Sull'integrazione dell'equazione di Darboux-Riccati nella teoria delle curve a curvatura doppia. Rend. Circ. Mat. Palermo (2) **1**, 88–91 (1952).

The integration of the equations of Frenet-Serret can be reduced to the integration of a Riccati equation, but few cases are known in which the integration of this last equation can be given explicitly. The author in this paper shows how it can be done in the case of the equations of Frenet for cylindrical and conical helices. He makes use of a substitution which transforms a Riccati equation into a linear equation of the second order. *E. T. Davies.*

De Sloovere, H. Sur le nombre d'invariants distincts, fonctions de tenseurs, d'après la méthode de Lie et De Donder. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 131–135 (1952).

The author shows that by use of new variables, the derivatives introduced by De Donder and used by the author in part I [same Bull. (5) **37**, 583–598 (1951); these Rev. **13**, 493] can be replaced by ordinary partial derivatives. Further, it is shown that this replacement of derivatives is possible in the complete systems derived in the previous paper, without changing the coefficients of structure.

N. Coburn (Ann Arbor, Mich.).

Cantoni, Riccardo. Superfici rigate e cinematicamente coniugate. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **13(82)**, 427–441 (1949).

Wenn dp den minimalen Abstand darstellt zweier consekutiven Strahlen einer geradlinigen Fläche und $d\xi$ den Winkel zweier benachtbarten solcher minimalen Abstände, dann heisst $\rho=dp/d\xi$ den kinematischen Krümmungsradius (raggio cinematico) des Strahls. Berechnung von ρ für die Schraubenfläche und das Rotationshyperboloid. Anwendung auf die feste und die bewegliche Achsenfläche einer räumlichen Bewegung. *O. Bottema* (Delft).

Jonas, Hans. Die allgemeinen äquidistanten Transformationen der von Bianchi entdeckten isometrischen Paare Tschebyschefscher Netze. J. Reine Angew. Math. **189**, 207–237 (1952).

Let $\mathfrak{x}$ and $\mathfrak{x}$ be two surfaces for which

$$(1)\qquad ds^2=d\bar{s}^2=d\alpha^2+2\cos\omega d\alpha d\beta+d\beta^2,$$
$$L=\bar{L},\quad M=\epsilon\sin\omega=-\bar{M},\quad N=\bar{N}\quad(\epsilon=\pm1)$$

where L, M, N $(\bar{L}, \bar{M}, \bar{N})$ are coefficients of the second fundamental form of $\mathfrak{x}$ $(\bar{\mathfrak{x}})$. Then there is a transformation

$$(2)\qquad \text{a) } \mathfrak{x}_1=\mathfrak{x}+k\mathfrak{X},\qquad \text{b) } \bar{\mathfrak{x}}_1=\bar{\mathfrak{x}}+\bar{k}\bar{\mathfrak{X}}$$

with constant $k, \bar{k}$ which carries these surfaces in $\mathfrak{x}_1, \bar{\mathfrak{x}}_1$ with the property (1). The vector $\mathfrak{X}$ $(\bar{\mathfrak{X}})$ satisfies a set of differential equations. The constant k is a function of two parameters κ, n, $k=k(\kappa, n)$. Consider now two transformations (κ_1, n_1) and (κ_2, n_2) of $\mathfrak{x}$ according to (2a). Applying $(\kappa_2 n_2)$ on $\mathfrak{x}_1$ and (κ_1, n_1) on $\mathfrak{x}_2$ we obtain the same surface $\mathfrak{x}^*$. A similar statement holds for $\bar{\mathfrak{x}}$. We obtain the surface $\bar{\mathfrak{x}}^*$. Both surfaces $\mathfrak{x}^*$ and $\bar{\mathfrak{x}}^*$ are isometric. The skew quadrilaterals $\mathfrak{x}\mathfrak{x}_1\mathfrak{x}^*\mathfrak{x}_2$ and $\bar{\mathfrak{x}}\bar{\mathfrak{x}}_1\bar{\mathfrak{x}}^*\bar{\mathfrak{x}}_2$ are the "Bianchi quadrilaterals", i.e., they are of the type obtained by composition of Bäcklund transformations. All these statements are proved by a straightforward computation and applied in particular on the net of the asymptotic lines of a pseudospherical surface. This enumeration of results and problems is by no means an exhaustive one. *V. Hlavatý* (Bloomington, Ind.).

Van Bouchout, V. Les lignes hexagonales dans les réseaux de surfaces. Colloque de Géométrie Différentielle, Louvain, 1951, pp. 175–182. Georges Thone, Liège; Masson & Cie, Paris, 1951. 350 Belgian francs; 2450 French francs.

A study of the order of closure of a certain curvilinear hexagon about M associated with three one-parameter families of surfaces, one surface of each family through M. Let $f_i(x_1', x_1^2x^3)=C_i$ be the equations of the surfaces S_i. The principal result may be stated as follows. In order that the closure be of order equal to or greater than two then $J=|\partial f_i/\partial x^j|=0$. *V. G. Grove.*

Roşca, Radu M. Sur les congruences doublement cycliques. An. Acad. Repub. Pop. Române. Secţ. Şti. Mat. Fiz. Chim. Ser. A. **3**, 209–246 (1950). (Romanian. Russian and French summaries)

In euclidean space a congruence is said to be cyclic if it is harmonic to an orthogonal net O. In elliptic space Σ_3 a conjugate net is called a principal system if it is conjugate to the congruence and at a distance of $\pi/2$ from the net O. To a principal system corresponds a cyclic system (a family

of circles admitting orthogonal surfaces) in Σ_3. For this reason the classification of cyclic congruences in Σ_3 with respect to their principal systems leads to a type of cyclic congruence not existing in euclidian space, namely those congruences which admit but two principal systems. Such congruences K_2 are the subjects of this study. Theorems of permutability are found. A congruence K_3 is the axis-congruence of a sequence of Laplace of period four, and is an R-congruence. Congruences K_3 are considered also as being determined by the intersection of two nets O related by a transformation of Ribaucour. *V. G. Grove.*

Inzinger, Rudolf. Eine projektiv invariante Konfiguration von Linienelementen dritter Ordnung. Monatsh. Math. 56, 38–48 (1952).

It is known, from the reviewer's papers [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 22, 483–491 (1935)] that two plane curvilinear elements of the 3rd order, in a generic position, have two projective invariants. In the present paper, three projective invariants, related by one equation, are introduced in the following way. A (general) plane E_2 is individuated by its center T, its tangent at T, t, and the pencil F of the conics going through T, tangent to t, and having third-order contact at T; or else, by the projectivity (correlation in T, t) between the point-range t and the bundle T, subordinated by the polarity with respect to any conic of F, T and t corresponding to each other in this correlation. Let now T_1, T_2, T_3 be three points of the plane, and put $T_1T_2 \equiv t_3$, etc.; moreover, let E_{ik} be the E_2, individuated by its center T_i, tangent t_k, and the correlation in T_i, t_k. The transforms of T_3 for the correlations in T_1, t_2 and in T_2, t_1 are two lines, p_1 (through T_1) and p_2 (through T_2), which meet in Z; similarly, t_3 gives origin to two points P_2 (on t_2) and P_1 (on t_1), joined by a line z; finally, put $Z \cdot T_3 \equiv p_3$; $p_3 t_i \equiv Q_i$ $(i=1, 2, 3)$. We obtain in such a way a configuration, projectively invariant to the pair E_{12}, E_{21}, and whose invariants are the same as for the pair E_{12}, E_{21}; these invariants are the four cross-ratios: $J_i = (P_i Q_i T_{i+1} T_{i+2})$, related by equation $J_1 J_2 J_3 = -1$, and these are the only invariants. J_3, in particular, is the characteristic of the (unique) perspective collineation of center P_3, axis p_3, and transforming E_{12} into E_{21}. The same configuration, by circular substitutions on the figures 1, 2, 3, determines three pairs of E_2's: E_{12}, E_{21}; E_{23}, E_{32}; E_{31}, E_{13}; and their invariants depend only on their elements of the 2nd order. The above configuration determines also a conic, for which $T_1T_2T_3$ is a self-polar triangle, while Z and z are pole and polar; the polarity with respect to this conic is the only one leaving the configuration invariant, and interchanges the E_2's of each pair. In the following paragraphs, the author gives metric meanings of the invariants J_i, and applications to the affine properties of the configurations. *E. Bompiani.*

Sydler, J.-P. Une propriété des espaces osculateurs des courbes normales. Comment. Math. Helv. 26, 36–41 (1952).

Let E^s denote a linear s-space contained in projective n-space E^n. A set of $t+1$ subspaces E^s are called associated if every E^{n-s-1} which meets any t of them also meets the remaining one. A normal curve is an algebraic curve of order n in E^n which is not contained in any E^s $(s<n)$. The following theorem and its dual are proved: Let $k \equiv n \pmod 2$, $s \equiv 1 \pmod 2$; $0 \leqq t \leqq s$. Given $k+s+1$ points (1) $P_0, \cdots, P_{k+s}$ on a normal curve in E^n, construct the subspace E spanned by the points $P_0, \cdots, P_{t-1}$ and by the intersection of the osculating E^{n-1}'s of the points (1). Project the osculating E^{k-1}'s of the points (2) $P_t, \cdots, P_{k+s}$ from E into the E^{k+s-t} spanned by the points (2). Then the projections are associated E^{k-1}'s in E^{k+s-t}. [In a letter to the reviewer, the author formulates more precise versions of this theorem. In particular, he points out that $k+s \leqq n$ has to be assumed.] Two more theorems apply the above result to general rational curves. *P. Scherk* (Los Angeles, Calif.).

Corio, Arnaldo. Sulle sezioni piane per un punto di una superficie aventi ivi un cerchio iperosculatore. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 85, 301–311 (1951).

This follows a paper by de la Gournerie [J. Math. Pures Appl. (1) 20, 145–156 (1855)] in which are studied the normal sections of a surface at points at which the derivative of the curvature vanishes. The present author studies sections other than normal sections of a surface at points where the same property holds. He proves that given a surface Σ and an arbitrary point P on it there are ∞^1 plane sections of Σ for which the derivative of the curvature vanishes at P. Through any straight line passing through P there pass five planes possessing the required property. If the straight line is tangent to Σ at P, only one of the five planes is not coincident with the tangent plane to Σ at P. If on the other hand the straight line is normal to Σ at P, then of the five planes passing through it two coincide with the isotropic planes, and the remaining three coincide with those already considered by de la Gournerie.
E. T. Davies (Southampton).

Wong, Yung-Chow. Fields of isocline tangent planes along a curve in a Euclidean 4-space. Tôhoku Math. J. (2) 3, 322–329 (1951).

Une R-surface dans un espace euclidien R_4 à 4 dimensions est caractérisé par la propriété que ses plans tangents sont isoclines les uns aux autres (ce qui est possible de deux façons). On sait [Eisenhart, Amer. J. Math. 34, 215–236 (1912)] que cette propriété est équivalente à la suivante: l'ellipse de courbure en tout point A de la surface est cercle de centre A. Utilisant les méthodes d'Elie Cartan, l'A. donne une solution du problème de Cauchy pour l'existence des R-surfaces. Tout d'abord, le long de toute courbe de R_4 différente d'une droite, il existe pour chaque plan initial tangent en un point A_0, deux champs isoclines de plans tangents, un de chaque type. A chacun correspond une R-surface et en ce sens une R-surface dépend de deux fonctions arbitraires d'une variable. Quelques propriétés des champs isoclines de plans tangents le long d'une courbe sont données et il est montré que toute famille à 1 paramètre de plans isoclines de R_4 est soit tangente à une courbe, soit pourvue d'un point fixe commun. Une classification complète des familles à 1 paramètre de plans de R_4 termine le papier. *A. Lichnerowicz* (Paris).

Coşniţa, Cesar. Sur les courbes et les surfaces anallagmatiques. Disquisit. Math. Phys. 7, 25–120 (1948).

An algebraic curve [surface] Γ in euclidean plane [space] is called anallagmatic if a circle [sphere] C exists, the directrix of Γ, such that an inversion with respect to C transforms Γ into itself. Γ can be constructed as the envelope of a one-parametric family of circles [spheres] which are perpendicular to C. If Γ is of order 3 or 4, then the locus of their centers is a conic [quadric]. The following is adapted from the introduction of the present paper: Darboux a étudié les surfaces anallagmatiques et les courbes sphériques anallagmatiques en déduisant de celles-ci les anallagma-

tiques planes. Dans ce travail nous avons fait une étude directe de ces courbes. Les travaux de Darboux et des autres auteurs sont limités surtout aux anallagmatiques du 3ième et 4ième ordre. Nous avons étudié certains questions sur les anallagmatiques générales, comme le degré, les points à l'infini, cas de décomposition, foyers, focales, tangentes doubles, diverses manières de génération, etc. Pour les anallagmatiques du 3ième et 4ième ordre nous avons donné aussi des propriétés nouvelles, ou nous avons repris des théorèmes connus avec des démonstrations et des conséquences nouvelles. [Cf. Darboux's monograph, Mém. Soc. Sci. Phys. Nat. Bordeaux **8**, 291–350; **9**, 1–280 (1873); also Kohn, Encyklopädie Math. Wiss., Teubner, Leipzig, 1909, Bd. III, 2.1, pp. 549–559; Meyer, 1931, Bd. III, 2.2, pp. 1618–1629; Berzolari, 1933, Bd. III, 2.2, pp. 2063–2065. The family of the Γ's of order 3 or 4 in (x, y)-plane is invariant under the group of broken linear transformations of $z=x+iy$. This seems to indicate that Darboux's approach to these curves is the more natural one. The author remarks that, in general, two plane Γ's of order 3 or 4 cannot be transformed into each other by an inversion (pp. 85 ff). This follows at once from the observation that the same holds true of their systems of directrix circles (cf. pp. 37 and 44) or still simpler from the fact that these Γ's form an 8-dimensional manifold while the set of all the inversions is only 3-dimensional.] *P. Scherk* (Los Angeles, Calif.).

Müller, Hans Robert. Isometrische Drehvorgänge und Beltramische Verbiegungen im elliptischen Raum. Math. Nachr. **7**, 213–218 (1952).

Verfasser verwendet für die Drehungen des euklidischen Raumes um einen festen Punkt O die Formel $x'=\bar{P}xP$, wo P und $\bar{P}$ konjugierte genormte Quaternionen und x, x' Ortsvektoren sind. Der Parameterraum besitzt eine elliptische Metrik mit $P\bar{P}=\bar{P}P=0$ als absolutum Gebilde. Den Bewegungen der dreigliedrigen Bewegungsgruppe werden nun die Punkte des elliptischen Raumes zugeordnet. Einem zwangsläufigen, bzw. einem flächenläufigen Drehvorgang entspricht also eine Kurve, bzw. eine Fläche in diesem Raum. "Isometrische Drehvorgänge" sind nun solche, die mit isometrischen Bildflächen korrespondieren. Verf. beweist noch für den elliptischen Raum den Satz von Beltrami: eine Strahlenkongruenz behält die Eigenschaft, Normalsystem zu sein, wenn die Strahlen bei Verbiegung einer beliebigen Schnittfläche mitgenommen werden, und er gibt von diesem Satz eine kinematische Deutung.

O. Bottema (Delft).

Cossu, Aldo. Trasformazioni puntuali tra spazi proiettivi osculabili con trasformazioni quadratiche. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **10**, 448–467 (1951).

Soit une transformation ponctuelle T entre deux espaces projectifs S_3, $\bar{S}_3$, envisagée dans le voisinage du 2ème ordre d'un couple régulier de points correspondants O, $\bar{O}$. La transformation T ne peut généralement pas être approximée, dans ce voisinage, par des transformations quadratiques, contrairement à ce qui a lieu pour les transformations ponctuelles entre deux plans, et il existe en général [Villa, mêmes Rend. (5) **3**, 216–230 (1942); ces Rev. **8**, 224] ∞^9 transformations cubiques osculatrices à T dans le voisinage du couple régulier O, $\bar{O}$. L'auteur considère la transformation générale rationnelle [2, 1] déterminée par le voisinage du 2ème ordre du couple O, $\bar{O}$ pour laquelle, aux droites issues de O dans S_3 correspondent les plans issus de $\bar{O}$ osculateurs aux E_2 correspondant, dans T, aux E_2 d'inflexion relatifs à ces droites. Il montre qu'on peut obtenir cette transformation en mettant en correspondance projective, les cônes d'un système déterminé de cônes cubiques de sommet O dont les éléments de base sont les droites caractéristiques [Borůvka] de la transformation, avec les droites de la gerbe de centre $\bar{O}$. En particularisant le système de cônes cubiques, on obtient les T qui peuvent être osculées par des transformations quadratiques des différentes espèces, générales et particulières. L'auteur montre ainsi que, pour chaque type T particulier il existe ∞^8 transformations quadratiques osculatrices, dont il donne les équations explicites relatives à des repères déterminés. Considérant ensuite le voisinage du 3ème ordre du couple O, $\bar{O}$, il détermine des repères projectifs intrinsèques pour chacun des cas envisagés.

P. Vincensini (Marseille).

Nitsche, Joachim. Bestimmung der Flächen, deren Bogenelement negativer Krümmung als Quadratsumme zweier Pfaffscher Formen gegeben ist. Arch. Math. **3**, 50–59 (1952).

Denote by M and G the Mainardi-Coddazi and Gauss equations of a surface. Write them in an orthogonal non-holonomic parameter system. Denote by h_{12} the middle non-holonomic component of the second fundamental tensor. Substituting from G into M for h_{12} one obtains a quasilinear system M' of two differential equations. This system is hyperbolic if and only if the surface is of negative Gauss curvature, which we shall assume. If one introduces the angle α, β of one of the non-holonomic directions (i.e., the null direction of the corresponding Pfaff form), then M' may be transformed into a system M^* of quasilinear differential equations for derivatives $\alpha_{(i)}$, $\beta_{(i)}$ ((i) denotes the derivative with respect to the arcs of the ith asymptotic line, $i=1, 2$). M^* is a quasilinear system in the sense of Haack and Hellwig [Math. Z. **53**, 244–266, 340–356 (1950); these Rev. **12**, 614]. Using the results of these authors one sees that M^* leads to a set of existence theorems. As a sample we quote at least one (in its rough form). Let $(u(s), v(s))$ be a curve in the (u, v)-plane and $\mathbf{x}(s)$ a solid curve. There are in general two surfaces $\mathbf{y}(u, v)$ with the given ds so that $\mathbf{y}(u(s), v(s))=\mathbf{x}(s)$. The system M^* is obviously particularly adapted to treat the existence theorems of minimal surfaces. *V. Hlavatý* (Bloomington, Ind.).

Suguri, Tsuneo. Theory of curves in the unitary K_n-connected spaces. Mem. Fac. Sci. Kyūsyū Univ. A. **6**, 31–40 (1951).

The author studies the differential geometry of curves (with a real parameter) in complex n-space with a positive definite Hermitian metric. *S. B. Myers.*

Adati, Tyuzi. On subprojective spaces. I. Tôhoku Math. J. (2) **3**, 159–173 (1951).

The author studies some of the relationships between subprojective spaces (defined by B. Kagan) and torse-forming and concircular vector fields (defined by K. Yano). It is shown that a Riemann space whose Christoffel symbols of the second kind may be written as $\{^{\lambda}_{\mu\nu}\}=\phi_\mu\delta_\nu^\lambda+\phi_\nu\delta_\mu^\lambda+\phi_{\mu\nu}\xi^\lambda$ where ϕ_μ, $\phi_{\mu\nu}$ are any covariant vector and tensor and ξ^λ is a torse-forming vector field is subprojective. A converse is also proved. Concerning concircular vector fields, the following is a typical result: In a Riemann space V_n $(n>3)$ admitting a concircular vector field, there exists a family of ∞^1 totally umbilical hypersurfaces whose normals are expressible by the vector field. If all these hypersurfaces are

of constant curvature, it is necessary and sufficient that V_n be conformally flat. *A. Fialkow* (Brooklyn, N. Y.).

Ščerbakov, R. N. Affine- and projective-invariant classes of lines on a surface characterized by means of an adjoined line. Doklady Akad. Nauk SSSR (N.S.) **83**, 39–42 (1952). (Russian)

The problem of adjoined classes of lines on two surfaces was first considered in its metric phase by Touganov [C. R. (Doklady) Acad. Sci. URSS **30**, 383–385 (1941); these Rev. **3**, 17]. The author of the present paper considers the affine and projective phases of the problem. If r, e_1, e_2, e_3 are the radius vectors of a point of a surface and the e's are the vectors of the moving trihedral, their derivatives are expressible linearly in terms of e_1, e_2, e_3 and five affine invariants α, β, γ and μ and π. The requirement that L^* of the second surface be invariantly associated with the line L of the first is met by the condition $a_0{}^3(a_1{}^1\alpha+a_1{}^3\gamma)+a_0{}^3(a_1{}^3\mu-a_1{}^1\pi)+a_1{}^2=0$, where the a's are constants. The paper is devoted to the examination of surfaces that satisfy special forms of the above relation. *M. S. Knebelman* (Pullman, Wash.).

Yanenko, N. N. On a connection between metric and projective properties of surfaces. Doklady Akad. Nauk SSSR (N.S.) **82**, 685–688 (1952). (Russian)

Let "a plane E^k" mean a k-dimensional linear subspace of the euclidean space E^{m+q}. The author has shown previously [same Doklady **65**, 449–452 (1949); **72**, 857–859, 1025–1028 (1950); these Rev. **11**, 396; **12**, 357] that if an m-dimensional manifold V^m in E^{m+q} possesses a proper isometric deformation, then it can be decomposed into ∞^r plane generators E^{m-r} along which the q-normal to V^m is constant. Then the additional result is proved that the plane E^r normal to E^{m-r} in the tangent plane E^m to V^m is spanned by planes $E^{\rho_1}, \cdots, E^{\rho_p}$, $\rho_1+\rho_2+\cdots+\rho_p=r$ and that each direction in E^{ρ_i} is focal and any plane formed by focal direction is focal. Interesting consequences of this result are: the class of all surfaces in E^{m+q} which possess (in the small) infinitesimal rigid deformations is projectively invariant (in E^3 this is trivial since every surface admits infinitesimal deformations). If a manifold V^m in E^{m+2} possesses infinitesimal deformations and is intrinsically congruent to an extrinsically not congruent manifold, then V^m can be rigidly deformed in a continuous way. *H. Busemann.*

Yanenko, N. N. On the class of a Riemannian metric. Doklady Akad. Nauk SSSR (N.S.) **83**, 533–536 (1952). (Russian)

The imbedding of a Riemann space V_n in a Euclidean E_{n+q} leads to the classical equations of Gauss and of Codazzi and Ricci. The author attacks the problem by means of exterior differential forms and solves the milder problem of obtaining criteria for class $\leqq q$ for metrics of type $t \geqq 3$. The presentation is very much condensed so that some forms are not defined and some definitions are not clear. *M. S. Knebelman* (Pullman, Wash.).

Yanenko, N. N. Metrics of class 2. Doklady Akad. Nauk SSSR (N.S.) **83**, 667–669 (1952). (Russian)

In this paper the author considers the criteria for a metric to be of class $q \leqq 2$ and type $t \leqq 2$ [cf. the preceding review]. If the given metric is $ds^2=g_{ij}\omega^i\omega^j$ where the ω's are linear differential forms in $du^1, \cdots, du^m$, the main theorem is then that if the type is 2, rank >5, and class $\leqq 2$, the metric $dS^2=g_{ij}\bar{\Omega}^i\bar{\Omega}^j+\beta^2\psi^2$ where $\bar{\Omega}^i=\omega^i+\delta_3{}^i dt+t\omega_3{}^i$ must be of class $\leqq 1$. *M. S. Knebelman* (Pullman, Wash.).

Pu, P. M. Some inequalities in certain nonorientable Riemannian manifolds. Pacific J. Math. **2**, 55–71 (1952).

Soit M^2 une variété différentiable compacte à deux dimensions admettant une famille de courbes fermées non homotopes à zéro. Si M^2 est muni d'une structure de variété riemannienne par la donnée d'une métrique, soient A l'aire de la variété et a la plus grande borne inférieure de la longueur de telles courbes. Dans le cas où M^2 est un tore, on sait, par exemple, d'après Loewner (non publié) qu'entre A et a, on a l'inégalité $A \geqq 3^{1/2}a^2/2$, quelle que soit la métrique envisagée et qu'il existe une métrique telle que l'égalité soit atteinte. Dans le présent papier, l'A. démontre l'existence d'inégalités analogues pour le plan projectif P^2 et la bande de Möbius. Dans le cas du plan projectif, l'A. adopte pour métrique privilégiée sur P^2 une métrique elliptique pour laquelle $A=(2/\pi)a^2$. Il établit à l'aide d'un lemme général qu'en changeant de métrique A ne peut que croître et a décroître. On a par suite pour toute métrique sur P^2, l'inégalité $A \geqq (2/\pi)a^2$, la constante $(2/\pi)$ étant la meilleure possible. Ce résultat est étendu, avec quelques restrictions et une constante convenable, au plan projectif P^n par utilisation de l'inégalité de Hölder. Dans le cas de la surface de Möbius, l'A. détermine encore, par intégration invariante et d'une manière fort intéressante, une métrique privilégiée

$$ds^2=\frac{4e^{\pi y/\alpha}}{(1+e^{\pi y/\alpha})^2}(dx^2+dy^2),$$

métrique pour laquelle on a la propriété:

$$A=\frac{2}{\pi}\frac{e^{\beta\pi/\alpha}-1}{e^{\beta\pi/\alpha}+1}a^2$$

où 2α et 2β sont les longueurs euclidiennes des côtés du rectangle fondamental définissant la surface. Il en déduit une inégalité analogue à celle valable pour P^2.

A. Lichnerowicz (Paris).

Sasaki, Shigeo. An alternative proof of Liber's theorem. Proc. Japan Acad. **27**, 73–80 (1951).

The title indicates the content of the paper. The theorems proved are slightly more general than those obtained by Liber [Doklady Akad. Nauk SSSR **66**, 1045–1046 (1949); these Rev. **11**, 134]. *M. S. Knebelman.*

Yano, Kentaro, et Sasaki, Shigeo. Sur la structure des espaces de Riemann dont le groupe d'holonomie fixe un plan à un nombre quelconque de dimensions. Proc. Japan Acad. **24**, no. 7–8, 7–13 (1948).

The paper generalizes earlier results of Sasaki (in Japanese) on spaces whose holonomy group fixes a point or a direction. It is proved that if the holonomy group of a Riemann space V_n fixes a line then it is necessary and sufficient that a coordinate system exist in which the first fundamental form of the space can be written

$$ds^2=(dx^1)^2+(dx^2)^2+(x^2)^2g_{pq}(x^r)dx^pdx^q,$$

$p, q, r=3, 4, \cdots, n$. An equivalent characterization of V_n shows that V_n must contain a family of ∞^1 totally geodesic hypersurfaces which have additional special properties. The structure of Riemann spaces whose holonomy group fixes two points is identical with that described above. These results are generalized to the cases in which the holonomy group fixes an m-dimensional plane or $m+1$ points. These two cases are different if $m>1$. *A. Fialkow.*

Wakakuwa, Hidekiyo. On Riemann spaces whose homogeneous holonomy groups are integrable. Tôhoku Math. J. (2) **4**, 96–98 (1952).

The paper generalizes two theorems due to Liber and to Kurita dealing with the decomposition of Riemann spaces whose homogeneous holonomy group is either of one (Liber) or of two (Kurita) parameters. The author examines Riemann spaces whose homogeneous holonomy group is of r parameters and is integrable. He proves that a V_n with such a group is the direct product of r two-dimensional Riemann spaces and a Euclidean E_{n-2r}. An obvious corollary is that if the group is integrable, $r \leqq [\frac{1}{2}n]$. *M. S. Knebelman.*

Kurita, Minoru. Riemann space with two-parametric homogeneous holonomy group. Nagoya Math. J. **4**, 35–42 (1952).

By means of five lemmas dealing with the rank of pfaffian matrices the author shows that if the homogeneous holonomy group of an n-dimensional Riemann space is of two parameters, the space is a direct sum of two non-flat two-dimensional Riemann spaces and an $(n-4)$-dimensional flat space. *M. S. Knebelman* (Pullman, Wash.).

Patterson, E. M. Some theorems on Ricci-recurrent spaces. J. London Math. Soc. **27**, 287–295 (1952).

Un espace riemannien, de signature quelconque, est dit Ricci-récurrent et est désigné par R_n si le tenseur de Ricci $R_{\alpha\beta}$ satisfait à

$$\nabla_\gamma R_{\alpha\beta} = R_{\alpha\beta} K_\gamma \quad (K_\gamma \neq 0).$$

On suppose $n>2$ et $R_{\alpha\beta} \neq 0$ pour écarter des cas singuliers. Un R_n admet un champ parallèle non trivial de plans. L'A. établit d'abord un théorème de réductibilité: Un R_n localement réductible est le produit d'un espace d'Einstein (avec $R_{\alpha\beta}=0$) par un $R_{n'}$ localement irréductible ou un V_2 à courbure non constante. Dans un R_n localement irréductible, les invariants principaux de Ricci sont tous nuls. A l'aide de ce résultat, l'A. établit que, pour un tel espace, si Λ est l'espace vectoriel des vecteurs λ^β tels que $R_{\alpha\beta}\lambda^\beta = 0$ et Λ' l'espace vectoriel orthogonal, Λ' est contenu dans Λ. Dans le cas $n=2p$, l'A. obtient le joli résultat suivant: pour $p>2$, un R_{2p} localement irréductible pour lequel le tenseur de Ricci est de rang p, est une extension riemannienne [Patterson et Walker, Quart. J. Math., Oxford Ser. (2) **3**, 19–28 (1952); ces Rev. **13**, 985] d'un espace de Weyl W_p dans lequel la partie symétrique du tenseur contracté de courbure peut être pris comme tenseur fondamental. Une réciproque est donnée. Les résultats sont étendus par des méthodes particulières au cas $p=2$. *A. Lichnerowicz* (Paris).

Yano, Kentarô Some remarks on tensor fields and curvature. Ann. of Math. (2) **55**, 328–347 (1952).

A côté de contributions personnelles intéressantes, ce papier contient un exposé systématique de certains résultats de S. Bochner [Ann. of Math. **49**, 379–390 (1948); ces Rev. **9**, 618] et de Lichnerowicz [C. R. Acad. Sci. Paris **226**, 1678–1680 (1948); Proc. Internat. Congress Math., Cambridge, Mass., 1950, v. 2, Providence, 1952, pp. 216–223; ces Rev. **9**, 618; **13**, 492]. Dans une première partie, l'A. explicite des formules intégrales valables pour toute forme extérieure sur une variété compacte et faisant intervenir soit les tenseurs de Riemann-Christoffel et Ricci, soit le tenseur

$$K^{(p)}_{ijkl} = \tfrac{1}{2}(R_{ik}g_{jl} - R_{jk}g_{il} + R_{jl}g_{ik} - R_{il}g_{jk}) + \tfrac{1}{2}(p-1)R_{ijkl}$$

introduit, à un facteur constant près, par le rapporteur. De ces formules, il déduit un certain nombre des théorèmes de Bochner et Lichnerowicz sur la non existence, sans certaines hypothèses, de formes harmoniques non triviales. Les démonstrations ne sont pas essentiellement différentes des démonstrations originales, mais leur groupement est fait d'une manière élégant. Dans une seconde partie, l'A. étudie les nombres de Betti des groupes semi-simples orientables compacts et retrouve par ces méthodes les résultats classiques $B_1 = B_2 = 0$, $B_3 \geqq 1$. A cet effet, l'A. utilise le résultat suivant: la métrique riemannienne naturelle est telle que pour tout bivecteur ξ^{ij}

$$-\tfrac{1}{4}\xi^{ij}\xi_{ij} \leqq R_{ijkl}\xi^{ij}\xi^{kl} \leqq 0.$$

Il en déduit en particulier que toute forme harmonique de degré 3 est à dérivée covariante nulle (ce qui est vrai quel que soit le degré de la forme, mais ne peut être obtenu par cette méthode). Des limitations précises des courbures de Weyl de l'espace de groupe sont aussi obtenues. La troisième partie est consacrée aux tenseurs de Killing, c'est-à-dire, aux tenseurs antisymétriques satisfaisant à

$$\nabla_j \xi_{ii_1 \cdots i_p} + \nabla_i \xi_{ji_1 \cdots i_p} = 0.$$

Sur l'existence de tels tenseurs des résultats symétriques de ceux concernant les tenseurs harmoniques sont systématiquement cherchés et obtenus. Le papier se termine par une caractérisation des tenseurs de Killing analogue à celle donnée par de Rham [de Rham et Kodaira, Harmonic integrals, Inst. Adv. Study, Princeton, 1950; ces Rev. **12**, 279] pour les tenseurs harmoniques. *A. Lichnerowicz.*

Libermann, Paulette. Sur les structures presque paracomplexes. C. R. Acad. Sci. Paris **234**, 2517–2519 (1952).

[This note is a sequel to Ehresmann and Libermann, same C. R. **232**, 1281–1283 (1951); Libermann, ibid. **233**, 17–19, 1571–1573 (1951); **234**, 1030–1032 (1952); these Rev. **12**, 749; **13**, 75, 780.] A paracomplex number is of the form $x+jy$ with $j^2=1$; Euclidean R^{2n} can then be considered as paracomplex n-space (this amounts to considering involutory automorphisms of R^{2n}). The author studies an infinitesimal structure in a manifold V_{2n}, which consists in having a paracomplex structure in the tangent spaces at all the points of V_{2n}. Using paracomplex-valued differential forms, the author discusses questions similar to those in the references above: integrability of the structure, its equivalence to absence of a torsion, defined by structure equations, (almost) parahermitian structure, induced affine connections, their curvature and torsion forms, relations between isotropy and local homogeneity. A specific structure which occurs as an exceptional case of restricted-isotropic structures is connected with the non-compact form of the exceptional group G_2. *H. Samelson* (Princeton, N. J.).

Kawaguchi, Akitsugu, and Katsurada, Yoshie. On areal spaces. IV. Connection parameters in an areal space of general type. Tensor N.S. **1**, 137–156 (1951).

This is a fourth paper which the senior of the two authors devotes to the problem of defining a metric space based upon a fundamental function $F(x, \partial x/\partial u)$ of the coordinates x in an n-dimensional space, and the derivative of the x with respect to the parameters u in an m-dimensional subspace. The spaces studied are given the general name of areal spaces. The main results of the preceding three papers of the series [Tensor N.S. **1**, 14–45 (1950), 67–88, 89–103 (1951); these Rev. **12**, 536; **13**, 384, 385] have been concerned with the restrictive conditions to be imposed upon the F in order that a two-index metric tensor suitable for the measurement of vectors can be deduced from the fundamental function F. This has led to the exhaustive study of

two particular classes of spaces called the metric (of which Riemannian space is one) and the submetric classes. In this paper an areal space of the most general type is studied without reference to the possibility of the existence of two-index metric tensors. Most of the paper is concerned with the case $m=2$. The F in this case provides a metric bitensor $g_{ij,kl}$ but not a metric tensor g_{ij}. The authors define a line-metric connection and an area-metric connection in this areal space. In §3 some results are given for arbitrary m.

E. T. Davies (Southampton).

Spencer, D. C. Cauchy's formula on Kähler manifolds. Proc. Nat. Acad. Sci. U. S. A. **38**, 76–80 (1952).

Le but de ce papier est de rechercher une formule de Cauchy pour les formes analytiques d'une variété kählerienne; il s'agit naturellement d'exprimer la valeur de la forme en un point d'un sous-domaine donné de la variété à partir d'une intégrale sur le bord de ce domaine, l'intégrant ne faisant intervenir que les valeurs de la forme sur le bord et des noyaux indépendants du sous-domaine considéré. Dans une première partie, l'A. rappelle la définition des opérateurs fondamentaux sur une variété kählerienne selon Garabedian et Spencer [Technical Report 17, Stanford University, 1951]. Des considérations analogues ont été aussi faites par le rapporteur [C. R. Acad. Sci. Paris **231**, 1413–1415 (1950); **232**, 146–147, 677–679 (1951); Colloque de Géométrie Différentielle, Louvain, 1951, Thone, Liège, 1951, pp. 99–122; ces Rev. **12**, 535, 536, 746; **13**, 688] et par Hodge [Proc. Cambridge Philos. Soc. **47**, 504–517 (1951); ces Rev. **13**, 75]. L'A. montre ensuite qu'une singularité fondamentale $\theta_p(\zeta, \phi)$ pour l'opérateur $\square$ existe sur toute sous-variété propre compact d'une variété kählerienne pour les types purs $(p, 0)$ et $(0, \sigma)$. A partir de telles singularités, l'existence d'une formule de Cauchy est établie pour les formes analytiques. [Note du rapporteur: la note de Guggenheimer donnée en référence est erronée comme Hodge et le rapporteur l'on signalé.]

A. Lichnerowicz (Paris).

Lichnerowicz, André. Variétés pseudokählériennes à courbure de Ricci non nulle; application aux domaines bornés homogènes de C^r. C. R. Acad. Sci. Paris **235**, 12–14 (1952).

It is shown that the restricted homogeneous holonomic group of a pseudo-Kählerian variety V_{2n} has a centre which is not discrete if its Ricci curvature is different from zero. It is further shown that if V_{2n} is a homogeneous space $V_{2n}=G/H$, where H is compact and $n>1$, and if the Kählerian metric of V_{2n} is invariant under G and has non-zero Ricci curvature, then V_{2n} is symmetric, provided that the linear group of isotropy of V_{2n} is irreducible.

W. V. D. Hodge (Cambridge, England).

Kuiper, N. H. On the holonomic groups of the vector displacement in Riemannian spaces. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 445–451 (1951); erratum: same Proc. Ser. A. **55** = Indagationes Math. **14**, 191 (1952).

Ce papier est consacré à l'étude du groupe d'holonomie homogène restreint σ_* (dans la terminologie du rapporteur) associé à un morceau de variété riemannienne. Le point du vue est strictement local. L'A. retrouve un certain nombre de résultats de Elie Cartan, en particulier sur la structure d'un groupe d'holonomie irréductible; il montre qu'un groupe irréductible ne peut jamais admettre un sous-groupe invariant de dimension 2. Il étudie sommairement le cas où σ_* irréductible admet un centre non discret; la variété est alors pseudo-kählerienne au sens du rapporteur. Celui-ci a établi que inversment le groupe σ_* de toute variété pseudo-kählerienne à courbure de Ricci différente de zéro admet un centre non discret [voir l'analyse ci-dessus]. Des considérations précédentes, l'A. déduit quelques résultats sur les dimensions possibles d'un groupe d' holonomie irréductible. Dans deux notes récentes [mêmes C. R. **234**, 1835–1837, 2332–2334 (1952); ces Rev. **13**, 986], A. Borel et le rapporteur ont donné des résultats beaucoup plus complets et rigoreux, les groupes d'holonomie d'une variété riemannienne étant définis et étudiés du point de vue global.

A. Lichnerowicz (Paris).

✓ ***Bochner, S. Laplace operator on manifolds.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 189–201. Amer. Math. Soc., Providence, R. I., 1952.

Dans cette conférence consacrée aux opérateurs self-adjoints L de type elliptique sur une variété différentiable V_k compacte (en général), l'A. a réuni un certain nombre d'applications à des domaines très variés de cet important instrument. I. Applications à la géométrie différentielle. Le lemme fondamental est le suivant: Si une fonction scalaire $f(x)$ sur V_k compacte satisfait à $Lf \geqq 0$, f se réduit à une constante; ce lemme a été donné sans hypothèse de self-adjunction et appliqué sous cette forme à des problèmes globaux de relativité en 1938 par le rapporteur [C. R. Acad. Sci. Paris **206**, 313–315 (1938)]. Les applications données explicitement ici sont les résultats de Bochner sur l'existence de champs de vecteurs de Killing ou de champs harmoniques, selon le signe de la courbure de Ricci de V_k douée d'une structure de variété riemannienne. Les résultats maintenant classiques du type courbure et nombres de Betti sont signalés. II. V_k étant supposée compacte et à structure analytique réelle, théorème de Bochner sur l'existence d'un homéomorphisme analytique plongeant V_k dans l'espace euclidien E_{2k+1}, dans l'hypothèse où V_k admet une métrique riemannienne analytique. Le raisonnement utilise essentiellement les fonctions propres du laplacien. III. Théorèmes stochastiques relatifs à l'"équation de la chaleur" $L_x f(t; x) = \partial f/\partial t$ sur V_k. Théorèmes de P. Lévy [Théorie de l'addition des variables aléatoires, Hermann, Paris, 1937, chap. VII] et Bochner [Ann. of Math. **48**, 1014–1061 (1947); ces Rev. **9**, 193]. IV. Théorèmes de quasi-analyticité, V_k et L sont supposés de classe C^∞. Toute fonction continue $f(x)$ admet un développement $\sum f_r(x)$ $(r=0, 1, \cdots, \infty)$ où f_r est fonction propre de L. Si

$$\sum_{n=0}^{\infty} \max_x |L^n f(x)|^{-1/n} = \infty$$

et si pour un point $x=\xi$, on a $L^n f(\xi)=0$, alors $f_r(\xi)=0$ quel que soit r. Application au tore et à l'opérateur $L=d/dx$. V. Dans le cas d'une V_{2k} à structure analytique complexe, non compacte, théorèmes de Widder et P. Lelong sur les fonctions indéfiniment dérivables dont les laplaciens successifs ont des signes alternés. VI et VII. Différentes applications dont la principale est la suivante: si V_{2k} est une variété kählerienne dont le tenseur de Ricci $R_{\alpha\beta}^*$ est défini négatif, si V_{2k} est recouvert par un nombre fini de voisinages U_α sur lesquels sont données des fonctions méromorphes ϕ_α dont les différences sont holomorphes dans $U_\alpha \cap U_\beta$, alors il existe une fonction Φ méromorphe sur V_{2k} telle que $(\Phi - \phi_\alpha)$ soit holomorphe dans U_α. Généralisation à k quelconque du caractère suffisant de la condition de Liouville pour l'existence de Φ (cas du multitore). VIII. Généralisa-

tions par Bochner [Ann. of Math. 44, 652–673 (1943); ces Rev. 5, 116] du théorème de Hartogs. *A. Lichnerowicz.*

Bochner, S. A new viewpoint in differential geometry. Canadian J. Math. 3, 460–470 (1951).

Soit V_n une variété différentiable, x et y deux systèmes de coordonnées d'un point $m \epsilon V_n$. Etant donné un tenseur T, admettant N composantes distinctes rangées dans un certain ordre, la loi de transformation de ces composantes dans le passage de l'un des systèmes de coordonnées à l'autre peut être décrite au moyen d'une matrice $a_B{}^A(y, x)$ $(A, B, \cdots = 1, \cdots, N)$ satisfaisant aux relations

$$(*) \qquad a_B{}^A(z, y)a_C{}^B(y, x) = a_C{}^A(z, x); \quad a_B{}^A(x, x) = \delta_B{}^A.$$

Cette matrice est reliée de la manière bien connue à la matrice jacobienne des x par rapport aux y et à la matrice inverse. L'auteur propose d'abandonner cette liaison et introduit, sous le nom de structure, un ensemble de matrices $a_B{}^A(y, x)$, défini dans l'intersection des domaines de deux systèmes de coordonnées et satisfaisant aux relations (*). L'interprétation en termes d'espaces fibrés sur V_n est claire et un "vectoroïde" apparaît comme une section, locale ou non, de l'espace fibré correspondant. Etant donnée la structure $a_B{}^A$, un vectoroïde (contravariant) $t^B(x)$ pour cette structure satisfait aux relations $t'^A(y) = a_B{}^A t^B(x)$. De la structure donnée on peut déduire une nouvelle structure $\bar{a}_B{}^A(y, x) = a_A{}^B(x, y)$ et les vectoroïdes pour cette structure seront dits covariants pour la structure initiale. Etant données plusieurs structures, distinctes ou non, on en déduit par produit kroneckerien une nouvelle structure et un vectoroïde pour cette structure sera dit un "tensoroïde" pour les structures initiales. L'auteur montre les avantages de ce point de vue, en particulier en ce qui concerne les tenseurs classiques présentant des symétries et montre comment on peut généraliser l'introduction d'un tenseur fondamental pour élever et abaisser les indices A. Dans une seconde section, l'auteur définit la notion de connexion affine de son point de vue; une telle connexion, pour une structure donnée, est définie par des quantités $L_{Ai}{}^B$ $(A, B = 1, \cdots, N; i = 1, \cdots, n)$ satisfaisant aux relations

$$a_A{}^B(y, x)\frac{\partial v^i}{\partial x^i}L'_{Bj}{}^C(y) = -\frac{\partial a_A{}^C(y, x)}{\partial x^i} + a_B{}^C(y, x)L_{Ai}{}^B(x).$$

Si $L_{Ai}{}^B$ est une connexion pour la structure $a_A{}^B$, alors $-L^B{}_{Ai}$ est une connexion pour la structure $\bar{a}_B{}^A$. L'auteur indique un certain nombre de propriétés de ses connexions qui généralisent au mieux les propriétés des connexions classiques. [Peut-être serait-il préférable pour ces définitions de raisonner plutôt en terme de différentielle absolue qu'en dérivée covariante, ce qui éviterait la dissymétrie de l'indice i.] La dernière section de cet intéressant papier est consacrée au cas où M_n est compacte; l'auteur y donne un théorème sur certains champs de vectoroïdes qui comprend, comme cas particuliers, des théorèmes antérieurement démontrés par lui [Bull. Amer. Math. Soc. 52, 776–797 (1946); Ann. of Math. 49, 379–390 (1948); ces Rev. 8, 230; 9, 618]: supposons donne sur M_n un tensoroïde symétrique positif S_{AB} et considérons les vectoroïdes t_A satisfaisant à

$$(**) \qquad g^{ij}t_{A,i,j} = S_{AB}t^B \quad (i, j = 1, \cdots, n).$$

Si S_{AB} est définie, (**) n'admet pas d'autre solution que 0; si S_{AB} est semidéfinie, les seules solutions de (**) sont telles que $t_{A,i} = 0$ et $S_{AB}t^B = 0$. Le papier s'achève sur une extension analogue d'un théorème intéressant de l'auteur relatif aux champs holomorphes de vecteurs sur une variété à structure analytique complexe [Ann. of Math. 52, 642–649 (1950); ces Rev. 12, 283]. *A. Lichnerowicz* (Paris).

García, Godofredo. New methods in the absolute differential calculus of G. Ricci and T. Levi-Cività and in the theory of general relativity of A. Einstein. Actas Acad. Ci. Lima 15, 25–92 (1952). (Spanish)

The author obtains the basic formulae of tensor analysis by the use of reciprocal frames of reference. The paper contains nothing essentially new. *H. S. Ruse.*

NUMERICAL AND GRAPHICAL METHODS

Salzer, Herbert E., Zucker, Ruth, and Capuano, Ruth. Table of the zeros and weight factors of the first twenty Hermite polynomials. J. Research Nat. Bur. Standards 48, 111–116 (1952).

This paper is concerned with the quadrature formulae

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)dx \doteq \sum_{i=1}^{n} \alpha_i^{(n)} f(x_i^{(n)})$$

and

$$\int_{-\infty}^{\infty} F(x)dx \doteq \sum_{i=1}^{n} \beta_i^{(n)} F(x_i^{(n)})$$

in which $F(x)$ is supposed to behave like e^{-x^2} times a polynomial, and $\beta_i^{(n)} = \alpha_i^{(n)} \exp{(x_i^{(n)})^2}$. The main content of the paper is a table giving values of $x_i^{(n)}$ to 15 decimals, of $\alpha_i^{(n)}$ to 13 significant figures, and of $\beta_i^{(n)}$ to 13 decimals; each for $0 < i \leqq n \leqq 20$. We may note that $0.49 < \beta_i^{(n)} < 1.8$ for all values given. A short introduction indicates a few properties of the functions concerned and the method of computation and checking used. *J. C. P. Miller.*

Abramowitz, Milton. Tables of the functions.

$$\int_0^{\phi} \sin^{1/3} x\,dx \quad \text{and} \quad (4/3)\sin^{-4/3}\phi \int_0^{\phi} \sin^{1/3} x\,dx.$$

J. Research Nat. Bur. Standards 47, 288–290 (1951).

This paper gives 8 decimal values of the first integral of the title for $\phi = 0(1^\circ)90^\circ$. The second integral is given to 8 decimals for $\phi = 0(\frac{1}{2}^\circ)90^\circ$, then to 8 figures (mainly 7 decimals) for $\phi = 90^\circ(\frac{1}{2}^\circ)180^\circ$. *J. C. P. Miller.*

van Wijngaarden, A. Table of the integral

$$\int_0^1 \exp{(-v^{-2} - xv)}v^{-p}dv.$$

Computation Department, Mathematical Centre, Amsterdam, Rep. R 176, 6 pp. (1952).

Let $F_p(x)$ denote the integral in the title. Then $F_p(0)$ is essentially an incomplete gamma function, and it satisfies $2F_p(0) + (3-p)F_{p-3}(0) = e^{-1}$. Using these facts a table giving $F_p(0)$ to 6D for $p = -25(1)11$ has been made. Several methods of computing $F_p(x)$ (and of checking these computations) for general x are indicated. A table to 5D is given

for $p=0(1)10$, $x=0(.25)3.5(.5)14.5$. It is stated that the last decimal is not absolutely reliable. *J. Todd.*

***Šumyagskiĭ, B. M. Tablicy dlya rešeniya kubičeskih uravneniĭ metodom osnov. [Tables for the solution of cubic equations by the method of bases.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 135 pp. 3.90 rubles.

The cubic equation whose roots are tabulated is taken in the form: (1) $z^3+Az=A$. The general cubic, once it is reduced to the usual trinomial $y^3+py+q=0$, becomes (1) when $-qz/p$ is submitted for y. Equation (1) has the roots $z_1=-3$, $z_2=3/2$, $z_3=3/2$ when $A=-27/4$ and has three real roots for all A not exceeding this value, the roots tending to $-\infty$, 1, and $+\infty$ respectively as $A\to-\infty$. For $A>-27/4$ the equation has, besides the real root z_1, the two complex roots $z_{2,3}=(-\frac{1}{2}\pm i\alpha)z_1$. For $A>-27/4$ the tables give z_1 and α; for $A<-27/4$ the tables give z_1 and z_2. (Of course, $z_3=-z_1-z_2$.) The tables are arranged in four parts with equal intervals given sometimes to the A's and sometimes to the z's with a view to maintaining about 4 significant figures in the z's and α's by means of either direct or inverse linear interpolation. *D. H. Lehmer.*

Hartley, H. O. Tables for numerical integration at non-equidistant argument steps. Proc. Cambridge Philos. Soc. **48**, 435–442 (1952).

Soient $x_0, x_1, \cdots, x_n$ des valeurs de x rangées par ordre croissant. Il est possible de trouver une fonction croissante $\phi(t)$ telle que $x_i=\phi(i)$, $i=0, 1, \cdots, n$. Pour évaluer numériquement $\int_{x_0}^{x_n} f(t)dt$ on peut transformer en $\int_0^n f(\phi(t))\phi'(t)dt$ et utiliser la formule de Cotes. Pratiquement on prendra pour $\phi(t)$ un polynome de degré n ou plus. Alors $\phi'(i)=\sum a_{ij}x_j$. Les a_{ij} forment pour chaque valeur de n une matrice carrée. L'auteur donne les termes de cette matrice pour $n=3, 4, \cdots, 10$. L'évaluation de l'erreur est envisagée. Deux exemples sont traités. *J. Kuntzmann* (Grenoble).

***Grosch, H. R. J. The use of optimum interval mathematical tables.** Proceedings, Scientific Computation Forum, 1948, pp. 23–27. International Business Machines Corp., New York, N. Y., 1950.

***Krawitz, Eleanor. Punched card mathematical tables on standard IBM equipment.** Proceedings, Industrial Computation Seminar, September 1950, pp. 52–56. International Business Machines Corp., New York, N. Y., 1951.

These two papers both start by considering types of table on punched cards suitable for use with IBM calculating equipment: critical tables, tables requiring linear interpolation, and those requiring quadratic and higher interpolation, are considered briefly in turn. Both authors are led to discuss optimum interval tables; these are derived as follows. First a particular method or formula of interpolation is chosen, and an upper limit to permissible errors. The tabular intervals are then computed such that the error of interpolation just reaches, but does not exceed, this upper limit; these intervals vary in size throughout the table, and a table constructed with these intervals is called a "maximum interval table". Finally, other considerations are introduced, such as the need for rational interval lengths, the desire not to change interval too frequently, requirements of the special equipment used, etc. These lead to the choice of intervals, all less than or equal to the corresponding maximum interval, most convenient for the particular calculation in hand and equipment available. A table with these intervals is called an "optimum interval table".

The first paper (Grosch) gives some of the theory behind the determination of maximum intervals for interpolation with polynomials of various degrees, and considers practical considerations leading to choice of optimum intervals. The second paper (Krawitz) gives practical details of the application of interpolation methods on IBM equipment, with particular reference to second-order interpolation on the Type 602-A calculating punch. *J. C. P. Miller.*

Hammersley, J. M. The computation of sums of squares and products on a desk calculator. Biometrics **8**, 156–168 (1952).

The author studies systematically routines for computing the sums $S_{ij}=\sum_{\alpha=1}^{A} x_{i\alpha}x_{j\alpha}$ $(i, j=1, \cdots, n)$ by means of a desk calculator. It is assumed that in one "run" $[p, q, r, s]$ the sums S_{pr}, $S_{ps}+S_{qr}$, S_{qs} can be obtained. If a routine requires $\rho_0+\xi$ runs (where ρ_0 is the least number of runs required to determine and check the S_{ij}), then ξ is called the index of the efficiency of the computing routine. The computing routine is supplemented by a rectifying routine; the expected number η of runs in the rectifying routine measures its efficiency. The author proposes to judge computing routines by their indices $\{\xi, \eta\}$ and gives a chain routine for the case $n=3m+2$ (m integer). This leads him to the conjecture that a routine of optimum efficiency would be of the type $\{\xi, \eta\}=\{0, 2\log_\alpha n\}$ with $3\leqq\alpha\leqq10$.

E. Lukacs (Washington, D. C.).

Salehov, G. S. On the convergence of the process of tangent hyperbolas. Doklady Akad. Nauk SSSR (N.S.) **82**, 525–528 (1952). (Russian)

A known iteration for finding a zero x^* of a real or complex function P with three derivatives is defined by

$$x_{n+1}=x_n-\frac{2P'(x_n)P(x_n)}{2[P'(x_n)]^2-P''(x_n)P(x_n)}.$$

The author interprets x_{n+1} as the x-intercept of the hyperbola $y=(x+\alpha)(\beta x+\gamma)^{-1}$ having second-order contact with $y=P(x)$ at the point (x_n, y_n). Methods developed by Kantorovič [Uspehi Matem. Nauk **3**, no. 6(28), 89–185 (1948), p. 170; these Rev. **10**, 380] in connection with Newton's method are applied to give conditions for the existence of a unique zero x^* in a given region $|x-x_0|<$constant, and to bound the constant C in the known relation $|x_{n+1}-x^*|\leqq C|x_n-x^*|^2$ (all n). There are two numerical examples and ten references. *G. E. Forsythe.*

***Sheldon, John W. Numerical evaluation of integrals of the form $\int_\alpha^\beta f(x)g(x)\,dx$.** Proceedings, Industrial Computation Seminar, September 1950, pp. 74–77. International Business Machines Corp., New York, N. Y., 1951.

This interesting, brief paper deals with numerical evaluation of an integral $\int_\alpha^\beta f(x)g(x)dx$ in the form $\sum d_i f(x_i)$ in the following cases. (1) $f(x)$ depends on one or more parameters, and integrations are desired over a range of parameters for the same $g(x)$. (2) The integrand is such that $g(x)$ has a singularity within the range of integration, such that $g(x)$ may be integrated analytically and $\int g(x)dx$ is finite, whilst $f(x)$ has no such singularity. (3) $g(x)$ is periodic and varies rapidly. Methods for evaluating the d_i are discussed and numerical illustrations given. *J. C. P. Miller.*

Andreev, B. A. Calculations of the spatial distributions of potential fields and their application to prospecting geophysics. III. Izvestiya Akad. Nauk SSSR. Ser. Geofiz. 1952, no. 2, 22–30 (1952). (Russian)

The author considers a so-called "two-dimensional" case when the geophysical anomaly is generated by a cylindrical structure extending to infinity in one direction, lying parallel to the horizontal surface of the ground and whose cross-section is the same everywhere. In such a case the potential $U(x, z)$ at a depth z ($z>0$) is related to the observed profile $U(x, 0)$ on the ground $z=0$ by the equation

$$\pi U(x, 0) = z\int_{-\infty}^{\infty} [(x-u)^2+z^2]^{-1}U(u, z)du.$$

Using this equation as well as the Laplace equation, the author studies a practical method for the computation of $U(x, z)$ from the observed values of $U(x, 0)$, based on the successive approximations. [For parts I and II see same journal 11, 79–92 (1947); 13, 256–267 (1949); these Rev. 11, 108.] *E. Kogbetliantz.*

Fox, L. Escalator methods for latent roots. Quart. J. Mech. Appl. Math. 5, 178–190 (1952).

The forward escalator method introduced by J. Morris [The escalator method in engineering, Wiley, New York, 1947; these Rev. 9, 382] is a technique for the computation of the latent roots and vectors of a matrix in terms of those of a submatrix. The backward escalator enables the latent roots and vectors of a submatrix to be calculated in terms of those of the complete matrix. In the method proposed by Morris the order of the submatrix is increased or decreased by one at each step. In this paper the author discusses the latent roots and vectors of $A-\lambda B$, A and B square matrices of order n, not necessarily symmetric, and by the use of partitioned matrices obtains a formula for increasing or decreasing the order of the submatrix by an integer p.

B. Friedman (New York, N. Y.).

Forsythe, George E. Alternative derivations of Fox's escalator formulae for latent roots. Quart. J. Mech. Appl. Math. 5, 191–195 (1952).

The formulas for the backward and forward escalator method given by Fox [see the paper reviewed above] are obtained by a more consistent use of partitioned matrices.

B. Friedman (New York, N. Y.).

Rosser, J. B., Lanczos, C., Hestenes, M. R., and Karush, W. Separation of close eigenvalues of a real symmetric matrix. J. Research Nat. Bur. Standards 47, 291–297 (1951).

A real symmetric test matrix A was constructed, whose 8 eigenvalues λ_i are approximately 102.005, 102.000, 101.990, 100.000, 100.000, 0.010, 0.000, −102.005; the matrix, its design, the exact λ_i, and the exact eigenvectors v_i are given. The method of Lanczos [same J. 45, 255–282 (1950); these Rev. 13, 163] is described as best adapted to desk computation, and was thus applied to yield the λ_i to 7 and the v_i to 6 decimals (if $|v_i|=1$). There is a detailed account of the calculation, which took about 100 hours. The method of Hestenes and Karush [ibid. 47, 45–61 (1951); these Rev. 13, 283] is described as best adapted for machine computation, and was used with an IBM Card-Programmed Calculator to yield the λ_i to 1 part in 10^8 and the v_i to about 8 decimals. The calculation is described; probably 3 or 4 days would have been ample time, had the operator already known the machine.

The paper describes some modifications of the "power method" (i.e., of computing $Ax, A^2x, \cdots, A^Nx$, where x is some vector) which would suffice to yield all the λ_i and v_i. For example, if two eigenvalues dominate, for large N they are approximately the zeros of the polynomials

$$\begin{vmatrix} 1 & a_{21} & a_{31} \\ \lambda & a_{22} & a_{32} \\ \lambda^2 & a_{23} & a_{33} \end{vmatrix},$$

where $a_{ij}=(y, A^{N+i-1}x)$, $a_{3j}=(z, A^{N+j-1}x)$, and y, z are vectors; a similar determinantal polynomial of degree k will yield k dominant eigenvalues. Use of Chebyshev polynomials in A and orthogonalization of the A^ix with respect to already known invariant spaces is also mentioned.

G. E. Forsythe (Los Angeles, Calif.).

Aržanyh, I. S. Extension of A. N. Krylov's method to polynomial matrices. Doklady Akad. Nauk SSSR (N.S.) 81, 749–752 (1951). (Russian)

The problem is to find the characteristic equation

$$|\Phi(\lambda)| = \sum_{i=0}^{sn} \alpha_i\lambda^{sn-i} = 0,$$

where $\Phi(\lambda)=\lambda^sI-\lambda^{s-1}A_1-\cdots-\lambda A_{s-1}-A_s$ is a polynomial matrix whose coefficients A_s are $n\times n$ matrices. In a direct application of Krylov's method, described for the ordinary case $s=1$ by H. Wayland [Quart. Appl. Math. 2, 277–306 (1945); these Rev. 6, 218], the α_i are determined from the linear dependence among the sn-rowed vectors $x_0, Ax_0, \cdots, A^{sn}x_0$, where

$$A=\begin{bmatrix} A_1 & A_2 & \cdots & A_{s-1} & A_s \\ I & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}.$$

A different algorithm, too formidable to reproduce, is proposed here, based on the author's (unavailable) generalization of the Hamilton-Cayley theorem [Doklady Akad. Nauk Uzbek. SSR 1951, no. 7], and involving iterations on s-tuples of n-rowed vectors. Calculating the vibrations of material systems with gyroscopic and dissipative forces furnishes an application of the case $s=2$. *G. E. Forsythe.*

Unger, H. Zur Auflösung umfangreicher linearer Gleichungssysteme. Z. Angew. Math. Mech. 32, 1–9 (1952). (English, French, and Russian summaries)

The author advocates solving a large linear system $\mathfrak{A}\mathfrak{x}=\mathfrak{a}$ (or inverting $\mathfrak{A}$) by first partitioning $\mathfrak{A}$ into a square matrix of submatrices $\mathfrak{A}_{lr}^{(1)}$ ($l, r=1, \cdots, p$), and then applying the ordinary Gaussian elimination to the partitioned system with coefficients $\mathfrak{A}_{lr}^{(1)}$. The first stage of this elimination is getting $\mathfrak{A}_{lr}^{(2)}=\mathfrak{A}_{lr}^{(1)}-\mathfrak{A}_{l1}^{(1)}(\mathfrak{A}_{11}^{(1)})^{-1}\mathfrak{A}_{1r}^{(1)}$ ($l, r=2, \cdots, p$). After the triangular resolution of $\mathfrak{A}_{11}^{(1)}$ has been obtained and checked, the matrices $\mathfrak{A}_{l1}^{(1)}(\mathfrak{A}_{11}^{(1)})^{-1}\mathfrak{A}_{1r}^{(1)}$ may be obtained and checked simultaneously by several persons. The "modernized Gaussian algorithm" of Cholesky and Banachiewicz is recommended for both types of computing steps. A detailed count shows that the partitioning scarcely affects the total number of multiplications, but shows how much the elapsed time can be reduced if there are q persons computing concurrently. There is a bibliography of 22 titles.

G. E. Forsythe (Los Angeles, Calif.).

*Herget, Paul. **Integration of the differential equation** $\frac{d^2P}{dr^2}=P\cdot F(r)$ **using the Type 601 Multiplying Punch.** Proceedings, Scientific Computation Forum, 1948, pp. 39–41. International Business Machines Corp., New York, N. Y., 1950.

*Lindberger, N. Arne. **Integration of second order linear differential equations on the Type 602 Calculating Punch.** Proceedings, Scientific Computation Forum, 1948, pp. 34–38. International Business Machines Corp., New York, N. Y., 1950.

These two papers are both concerned with practical details of the solution of a second-order linear differential equation, with first derivative absent, on IBM equipment. *J. C. P. Miller* (Cambridge, England).

*Johnson, Walter H. **Numerical solution of two simultaneous second-order differential equations.** Proceedings, Industrial Computation Seminar, September 1950, pp. 71–73. International Business Machines Corp., New York, N. Y., 1951.

This discusses two simultaneous second-order linear differential equations and their approximation in terms of difference equations, with some details of their solution on the IBM Type 604 Calculating Punch. *J. C. P. Miller.*

Chatterji, L. D. Radial oscillations of a gaseous star of polytropic index 1. Proc. Nat. Inst. Sci. India 17, 467–470 (1951).

For polytropic index $n=1$ Emden's equation is

$$\frac{d^2u}{dz^2}+\frac{2}{z}\frac{du}{dz}+u=0,$$

with solution $u=(\sin z)/z$. Small adiabatic radial oscillations in this model are derived from the equation

$$\frac{d^2f}{dz^2}+\frac{2(1+z\cot z)}{z}\frac{df}{dz}+\left\{\frac{\omega^2 z}{\sin z}+\frac{2\alpha(z\cot z-1)}{z^2}\right\}f=0,$$

where $\alpha=3-4/\gamma$, γ being the ratio of specific heats. For, the case $\alpha=0.6$, the characteristic values of ω^2 for the fundamental mode of vibration, and for the first and second overtones have been obtained to 3 decimals, and the values of f given to six decimals in each case, for $z=0(0.1)3.1$, the boundary $u=0$ being at $z=\pi$. The value of ω^2 for the fundamental mode is also given for $\alpha=0.4$ and $\alpha=0.2$. *J. C. P. Miller* (Cambridge, England).

*Panov, D. Yu. **Spravočnik po čislennomu rešeniyu differencial'nyh uravnenii v častnyh proizvodnyh. [Handbook on the numerical solution of partial differential equations.]** 4th ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 183 pp. 2.90 rubles.

This compact little book presents in brief form a quite complete introduction to the methods of solving partial differential equations by means of nets and associated difference equations. The introductory chapter shows how a two-dimensional region is covered by a net, usually rectangular, and how the differential equation (in this case Laplace's equation) is replaced by a difference equation. The second chapter develops the essential features of the difference calculus, presenting formulas, diagrams, and tables of coefficients for Newton's, Stirling's, Bessel's, and Lagrange's ($n=3$, $n=4$) methods of interpolation, and for Stirling's and Bessel's formulas for numerical differentiation. Difference formulas for partial derivatives are included. Chapter III deals with the choice of nets, the various difference formulas for Laplace's operator, solution of the finite linear system by iteration, accelerating convergence, formulas for corrections, and examples of the numerical solution of Dirichlet problems. Numerous figures and diagrams aid the explanations. In similar manner Chapter IV treats the biharmonic equation, V the heat equation, VI the wave and telegrapher's equations, and VII the quasi-linear hyperbolic system. The methods are well illustrated by worked examples. *W. E. Milne* (Corvallis, Ore.).

*Yowell, Everett C. **Numerical solution of partial differential equations.** Proceedings, Computation Seminar, December 1949, pp. 24–28. International Business Machines Corp., New York, N. Y., 1951.

The numerical solution of Laplace's equation on an IBM 604 electronic calculating punch is discussed. A modification of the Liebmann procedure due to W. E. Milne was used. *H. Polachek* (Silver Spring, Md.).

Wasow, Wolfgang. On the truncation error in the solution of Laplace's equation by finite differences. J. Research Nat. Bur. Standards **48**, 345–348 (1952).

The difference between the solutions of the Laplace differential equation and the Laplace difference equation, defined in the same rectangle and assuming the same boundary values, is estimated under the assumption that the boundary function possesses a bounded third derivative. The estimate depends on the square of the mesh length, on the numerical maximum of the second derivative of the given boundary function at the vertices of a rectangle, and on the numerical maximum of the third derivative of the boundary function anywhere on the rectangle. *B. Friedman.*

*Taub, A. H. **A sampling method for solving the equations of compressible flow in a permeable medium.** Proceedings of the Midwestern Conference on Fluid Dynamics, 1950, pp. 121–127. J. W. Edwards, Ann Arbor, Michigan, 1951.

"In this paper we investigate a single particle undergoing a Markov process and treat the transition probabilities of that process as the Maxwell-Boltzmann distribution function of a gas. We shall show that, if one then defines density, velocity, pressure, and heat flow as in kinetic theory, these quantities satisfy the partial differential equations describing the behavior of a fluid in a permeable medium. The transition probabilities of a Markov process may be calculated by a sampling (Monte Carlo) method with the aid of automatic computing machines. If this is done, then the results quoted above enable one to determine the behavior of fluids in a permeable medium without any reference to the partial-differential equations themselves. The errors involved are errors due to the size of the sample taken and may be readily estimated." (From the author's introduction.) *A. S. Householder* (Oak Ridge, Tenn.).

van den Dungen, F. H. Formules pour l'intégration numérique de l'équation des ondes. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 39–49 (1952).

The author proposes a method for the numerical solution of the following Cauchy problem: Find u such that (*) $\Delta u=c^{-2}u_{tt}$ given $u=f$, $u_t=g$ at $t=0$. By means of Taylor's series, the values of u and u_t for $t=h/c, 2h/c, \cdots$ can be approximated by the values of u and its time derivatives at $t=0$. With the help of (*) these latter values are expressed in terms of f, g, and their space derivatives. These space

derivatives are approximated by means of differences along a square mesh of width h, care being taken not to use points outside the domain of dependence of the solution. The author indicates that the formulas so obtained approach the exact solutions of the Cauchy problem as h approaches zero. *B. Friedman* (New York, N. Y.).

Bückner, Hans. Konvergenzuntersuchungen bei einem algebraischen Verfahren zur näherungsweisen Lösung von Integralgleichungen. Math. Nachr. **3**, 358–372 (1950).

Die Methode, Lösungen sowie Eigenwerte und auch Eigenfunktionen einer homogenen Integralgleichung mit Hilfe eines Systems linearer Gleichungen als finites Analogon der Integralgleichung näherungsweise zu bestimmen, wird hier als algebraisches Verfahren bezeichnet. Hilbert hat die homogene Integralgleichung

$$\varphi(s)-\lambda\int_0^1 K(s,t)\varphi(t)dt=0$$

ersetzt durch das Gleichungssystem

$$\varphi(kh)-\lambda\sum_{i=1}^{n}K(kh,ih)\varphi(ih)=0 \quad (k=1,2,\cdots,n),\quad h=\frac{1}{n},$$

und gezeigt, dass die Wurzeln der zugehörigen Determinante $d(\lambda)$ nach den Eigenwerten konvergieren für $n\to\infty$. Der Verf. gibt eine Verallgemeinerung dieses Verfahrens für beliebige Intervalleinteilung. Er untersucht weiter die Konvergenz und den Konvergenzgrad der Näherungslösungen. *S. C. van Veen* (Delft).

Kourganoff, Vladimir, et Pecker, Charlotte. Sur le choix des formules d'intégration numérique dans la résolution des équations intégro-différentielles de transfert (rayonnement, neutrons) par la "méthode des ordonnées discrètes". Ann. Astrophysique **12**, 247–263 (1949).

In the method of solving problems of radiative transfer in which integro-differential equations are replaced by systems of linear equations it has been customary to use the Gaussian formula for numerical quadratures [cf. Chandrasekhar, Radiative transfer, Oxford, 1950; these Rev. **13**, 136]. While all the formal developments are valid for any other quadrature formula, the particular advantage of the Gaussian formula is that all the exact relations of a problem are retained in all orders of approximation; other quadrature formulae do not have this property. However, on account of the logarithmic singularity in the derivative of the source function as we approach the boundary of the atmosphere, a quadrature formula other than Gauss' can give a better "overall accuracy". From a numerical analysis of the problem with a constant net flux in semi-infinite atmospheres, the authors show that the Tchebycheff and the Newton-Cotes quadrature formulae achieve this.

S. Chandrasekhar (Williams Bay, Wis.).

Vogt-Nilsen, Nils. A note on the electrolytic double-tank. Norske Vid. Selsk. Forh., Trondheim **24** (1951), 98–101 (1952).

ASTRONOMY

Bragard, L. La méthode dynamique en géodésie. Bull. Géodésique **1952**, 35–57 (1952). (French. German, English, Spanish and Italian summaries)

The author shows that the formula of Stokes [Trans. Cambridge Philos. Soc. **8**, 672–695 (1849)] expressing the relationship between the shape of a spheroid-approximating geoid and the force of gravity at all points on the surface, although originally derived with only the first order terms in the flattening considered, is actually rigorous.

R. G. Langebartel (Urbana, Ill.).

Bragard, Lucien. Sur une relation entre la densité d'une masse fluide et la pesanteur superficielle. C. R. Acad. Sci. Paris **234**, 2341–2343 (1952).

Use is made of different formulae for the potential of a fluid mass rotating with a constant angular velocity in order to show that the density at a point can be expressed in terms of the superficial gravity acceleration on the radius as well as in terms of the angular velocity and of the angle between the normal to the free surface and the radius.

W. Jardetzky (New York, N. Y.).

Agostinelli, Cataldo. Configurazioni di equilibrio di una massa liquida omogenea attratta da più centri lontani con la legge di Newton. Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (2) **71**, 1–48 (1951).

It is proved that under conditions mentioned in the title the free surface of a homogeneous liquid mass can have the form of an ellipsoid with three unequal axes. The liquid particles, then, have to move along elliptical paths. The case of a heterogeneous liquid was considered by the author in another paper [Univ. e Polytecnico Torino. Rend. Sem. Mat. **9**, 179–224 (1950); these Rev. **12**, 753].

W. Jardetzky (New York, N. Y.).

Dive, Pierre. De l'impossibilité d'une stratification ellipsoidale dans un fluide doué de rotations barotropes permanentes. Bull. Sci. Math. (2) **76**, 38–50 (1952).

The author reconsiders the question of an ellipsoidal stratification in a fluid mass and proves now that it is impossible if the rotation is a barotropic one. The reviewer [Zapiski Russk. Naučn. Inst. Belgrad **4**, 45–74 (1931)] has extended Volterra's proof for a constant angular velocity to the case treated in this paper. *W. Jardetzky.*

Jeffreys, Harold. Problems of thermal instability in a sphere. Monthly Not. Roy. Astr. Soc. Geophys. Suppl. **6**, 272–277 (1952).

L'auteur et Bland [même journal **6**, 148–158 (1951); ces Rev. **13**, 167] avaient trouvé antérieurement que dans la convection thermique d'une sphère fluide gravitante le mode d'instabilité le plus facile à exciter est celui où la perturbation de température est harmonique d'ordre un. La propriété n'est plus vraie si la sphère contient un noyau fluide isotherme de rayon moitié, ou si le mouvement du fluide se fait par percolation à travers un milieu spongieux. Mais ceci n'est pas une objection à la séparation ultime d'un hémisphère continental et d'un hémisphère océanique, car les circulations étendues peuvent détruire progressivement les perturbations de température qui causaient les circulations de dimensions moindres et d'énergie supérieure.

J. Coulomb (Paris).

***von Weizsäcker, Carl Friedrich. Anwendungen der Hydrodynamik auf Probleme der Kosmogonie.** Festschrift zur Feier des zweihundertjährigen Bestehens der Akademie der Wissenschaften in Göttingen. I. Math.-Phys. Kl., pp. 86–122. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1951.

This is a popular version of the author's more technical paper in Astrophys. J. 114, 165–186 (1951); these Rev. 13, 498. *S. Chandrasekhar* (Williams Bay, Wis.).

Lüst, Reimar. Die Entwicklung einer um einen Zentralkörper rotierenden Gasmasse. I. Lösungen der hydrodynamischen Gleichungen mit turbulenter Reibung. Z. Naturforschung **7a**, 87–98 (1952).

Von Weizsäcker [same Z. **3a**, 524–539 (1948); these Rev. 10, 488] has written down the equations governing a distribution of matter in which Reynolds stresses dominate the thermal pressure and molecular viscosity. In von Weizsäcker's equations the Reynolds stresses are evaluated in the usual manner in terms of a heuristically-defined coefficient of eddy viscosity $\eta=\delta\rho l^2 w$ where δ is a numerical factor of order unity, ρ is the density, l is the mixing length and

$$w=\left[\sum_{i,k}\left\{\left(\frac{\partial v_i}{\partial x_k}\right)^2+\frac{\partial v_i}{\partial x_k}\frac{\partial v_k}{\partial x_i}\right\}\right]^{1/2}.$$

In the special case when the distribution of matter has axial symmetry and is spread in the form of a thin "sheet" and in which the transverse velocity v_φ is determined by the gravitational potential $\Phi(s)$ (s is the distance from the axis of symmetry), by $v_\varphi^2=s\partial\Phi/\partial s$, and the radial component of the velocity, v_s of turbulent origin, is small compared to v_φ, von Weizsäcker's equations reduce to

$$\sigma v_s v_\varphi=+\frac{1}{s}\left[\eta s^2\left(\frac{v_\varphi}{s}\right)'\right]'\ ;\quad v_\varphi^2=s\Phi', \tag{1}$$

$$\dot\sigma=-\frac{1}{s}(\sigma s v_s)'=-\frac{j'}{s}, \tag{2}$$

$$w=s\left|\left(\frac{v_\varphi}{s}\right)'\right|\quad\text{and}\quad \eta=\delta\rho l^2 w, \tag{3}$$

where primes denote differentiation with respect to s and dots differentiation with respect to time. In the foregoing equations, σ denotes the surface density in the "sheet" and $j=\sigma s v_s$ is the material current density determining the rate of transport of matter across s per unit solid angle. In this paper the author considers these simplified equations in the case when the gravitational potential arises from a central mass M and $v_\varphi=(GM/s)^{1/2}$ (G denotes the constant of gravitation). With the further assumption that the mixing length $l=\alpha s^n$ where α is a constant, it is shown that the general solution for σ and j can be expressed as linear combinations of the fundamental solutions represented by

$$\begin{aligned}\sigma(s,t)&=s^{1/4-2n}J_{\pm\nu}(2\lambda s^{m/2}/m)e^{-\lambda^2 kt},\\ j(s,t)&=\mp k\lambda s^{m/2-1/4}J_{\pm(\nu-1)}(2\lambda s^{m/2}/m)e^{-\lambda^2 kt},\end{aligned} \tag{4}$$

where $\nu=1/2m$, $m=7/2-2n$ ($m\neq 0$), $k=9\delta\alpha^2(GM)^{1/2}/4$. In (4), $J_{\pm\nu}$ denotes the Bessel function of order $\pm\nu$ and λ is a characteristic value parameter determined by $J_{\pm\nu}(2\lambda s_1^{m/2}/m)=0$, where s_1 defines the extent of the system. By considering the behavior of the solution (4) for $s\to 0$ the author first shows that the exponent in the "law" $l=\alpha s^n$ should be restricted by $0\leqq n<5/4$; further, that the solution with $+\nu$ corresponds to a negative source of mass at the center (i.e., there is an inflow of matter here) while for the solution with $-\nu$ there is no such inflow since $j\to 0$ as $s\to 0$. The author also discusses the rate of change of the angular momentum $\sigma s v_\varphi$ given by

$$\frac{\partial}{\partial t}(\sigma s v_\varphi)=-\frac{1}{s}\left[\sigma s^2 v_s v_\varphi-\eta s^3\left(\frac{v_\varphi}{s}\right)'\right]'.$$

The transport of the angular momentum is therefore determined by

$$K=\sigma s^2 v_s v_\varphi-\eta s^3(v_\varphi/s)'=jsv_\varphi+\delta\sigma l^2 s^4(v_\varphi/s)'^2,$$

where the two terms on the right-hand side correspond to the transport by material velocity and turbulent viscosity, respectively. Evaluating K for the solution (4), the author shows that $K\to 0$ as $s\to 0$ for the solution with $+\nu$ while it tends to a finite limit for the solution with $-\nu$. Combining this result with the earlier one regarding j, we conclude that the possible solutions are of two types: Either the angular momentum of the central body remains constant and a part of the material in the envelope falls towards the center and the rest disperses to infinity, or the mass of the central body remains constant, the angular momentum of the body decreasing and the entire material in the envelope dispersing to infinity and carrying away the angular momentum. This is the principal result of the paper. Numerical illustrations of the theory are also given. *S. Chandrasekhar.*

Trefftz, Eleonore. Zur Entwicklung einer rotierenden Gasmasse. Z. Naturforschung **7a**, 99–103 (1952).

In this paper the author discusses the same problem as that investigated by Lüst in the paper reviewed above except that instead of assuming that the gravitational potential arises from a central point mass, the assumption is made that the system is a self-gravitating one. Thus in equations (1)–(3) of the preceding review, Φ is one of the variables of the problem to be determined from Poisson's equation. In this case an explicit solution cannot be found. Restricting herself to a two-dimensional model, the author integrates numerically the resulting partial differential equation for ρ for the case when the initial distribution of density is of the form $\rho=(1+s^2)^{-2}$. The results of the calculations are illustrated by a number of graphs which reveal the tendency of the system to evolve towards a state in which the matter in the inner parts tends to a uniform distribution while the matter in the outer parts disperses to infinity.

S. Chandrasekhar (Williams Bay, Wis.).

Chazy, Jean. L'intégration du problème des trois corps par Sundman, et ses conséquences. Bull. Astr. (2) **16**, 175–190 (1952).

Lecture given at Johns Hopkins University in September, 1950.

Tzénoff, I. Équilibre d'un point assujetti à se déplacer sur la surface d'un ellipsoïde homogène de révolution animé d'une rotation uniforme autour de son axe et attiré suivant la loi de Newton par les éléments de l'ellipsoïde. Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. **46**, 1–8 (1950). (Bulgarian. French summary)

The condition of equilibrium is

$$\omega^2=2\pi\rho f\frac{(3+\lambda^2)\operatorname{arctg}\lambda-3\lambda}{\lambda^3},$$

where ω is the angular velocity of rotation, ρ the density, f the gravitational constant, and λ the eccentricity of the meridian ellipse. For the earth, using $\rho=5.548$, the resulting

oblateness is 1/233.89. This value is too large, so the author concludes that, if the earth is an ellipsoid of revolution, it is not homogeneous. *L. Jacchia* (Cambridge, Mass.).

Popović, Božidar. New forms of the equations of perturbation in planetary motion. Glas Srpske Akad. Nauka. Od. Prirod.-Mat. Nauka **198**, 129–139 (1950). (Serbo-Croatian)

The author establishes a system of three differential equations which express the variations of the vectorial elements in function of the perturbing force. Formally, these equations differ from those previously obtained by Milankovitch and by Bilimovitch for the same problem because these authors used a force function instead of the perturbing force. When, however, the perturbing force is assumed to be the gradient of the force function, identity is achieved. The author's equations are of a more general character since no assumption is made concerning the perturbing force. *L. Jacchia* (Cambridge, Mass.).

Fricke, W. Dynamische Begründung der Geschwindigkeitsverteilung im Sternsystem. Astr. Nachr. **280**, 193–216 (1952).

In the past the Liouville equation has been used in stellar dynamics to enumerate the circumstances under which a dynamical system will admit an integral (other than the energy integral) which is quadratic in the velocities. And by postulating an integral of this form we restrict the gravitational potential in a certain way. Extending similar discussions by other writers, the author shows that the requirements of Liouville's equation and Poisson's equation cannot be met simultaneously by any reasonable physical system. The alternative approach recommended by the author is the following: The motion of a particle in an axially symmetric potential field $\Omega(R, z)$ admits the energy (E_1) and the angular momentum (E_2) integrals, $E_1=2\Omega-V^2$ and $E_2=Rv$ where V denotes the magnitude of the velocity and v its transverse component. The assumption is then made that the distribution function f in the phase space can be expressed in the form

$$f=f(E_1, E_2)=\sum\alpha_{ik}(2\Omega-V^2)^i(Rv)^k,$$

where α_{ik} are constants and the series on the right-hand side consists of a finite number of terms. The density distribution $\rho(R, z)$ corresponding to this form of f is given by

$$\rho(R, z)=\sum\alpha_{ik}\int\int\int_{0\leq V\leq\sqrt{(2\Omega)}}(2\Omega-V^2)^i(Rv)^k\,du\,dv\,dw.$$

The result of the integration is

$$\rho(R, z)=2\pi\sum\frac{\alpha_{i,2k}}{2k+1}B(k+3/2, i+1)R^{2k}(2\Omega)^{i+k+3/2},$$

where $B(x, y)$ denotes the beta-function $\Gamma(x)\Gamma(y)/\Gamma(x+y)$.

Now we know that for $r\to\infty$ the gravitational potential of a finite distribution of total mass M_1 must tend to M_1/r. Letting

$$\Omega=\frac{M_1}{r}+\frac{M_2(\mu)}{r^2},$$

where $\mu=\cos^{-1}(z/r)$ represents the asymptotic form of Ω, the author shows that Poisson's equation requires, first, that $i\geqq k+5/2$ and that we should be able to write

$$\rho(r, \mu)=2\pi\sum_{i=0}^{m}\sum_{k=0}^{n}\beta_{ik}r^{2k}(1-\mu^2)^k\Omega^{i+2k+1},$$

where the coefficients β_{ik} are related to the coefficients α_{ik} and, second, that $M_2(\mu)$ has the form

$$M_2(\mu)=E\mu-4\pi^2M_1^4\sum_{k=0}^{n}\frac{\beta_{0k}}{2k+1}\times\left[(1-\mu^2)^{2k}+2k\mu\int(1-\mu^2)^{2k-1}d\mu\right]$$

where E_0 is a constant of integration. Next, expanding $\Omega(r, \mu)$ at $r=0$ in the form $\Omega(r, \mu)=\sum_{n=0}^{\infty}\Omega_n(\mu)r^n$, the author shows how by using Poisson's equation each of the functions $\Omega_n(\mu)$ can be determined as simple polynomials in μ. From a comparison of these results with the known kinematical properties of the Milky Way system, the author concludes that a consistent galactic dynamics can be constructed along the lines of this paper. *S. Chandrasekhar.*

Stanyukovič, K. P. On the origin of the solar system. Akad. Nauk SSSR. Astr. Žurnal **29**, 288–305 (1952). (Russian)

The author proposes that the planets were formed from gaseous material that had been expelled with great force from the sun. Most of the gas particles would travel in hyperbolic orbits and be lost to the system but a small percentage of the orbits would be elliptic with the planets supposed formed from this material by condensation. The author treats the behavior of a stream of the expelled material on the basis of gas dynamics to show it is possible in this way to explain the observed distribution of angular momentum in the solar system. For the rest of the theory (which is sketched only qualitatively) a future quantitative investigation is promised. *R. G. Langebartel.*

Kushwaha, R. S., and Bhatnagar, P. L. Stability of stars under variable Γ. Proc. Nat. Inst. Sci. India **18**, 13–19 (1952).

Following the methods of a companion paper [Kushwaha, same Proc. **17**, 323–329 (1951); these Rev. **13**, 498] the authors treat the stability of the fundamental mode of the homogeneous model and the model with density falling off from the center inversely as the square of the distance, using as law of variation of Γ, the effective ratio of specific heats, $\Gamma=\Gamma_0(1-Ar_0^2/R^2)$, where r_0 is the distance from the center, R the star's radius, and Γ_0 is the value of Γ at the center. It is found that the variable density model is more stable than the homogeneous model. *R. G. Langebartel.*

Unno, Wasaburo. On the radiation pressure in a planetary nebula. I. Proc. Japan Acad. **26**, no. 6, 17–22 (1950).

This paper deals with essentially the same problem as treated by Miyamoto [Astrophys. J. **113**, 181–192 (1951); these Rev. **12**, 544] with a slight difference only in the physical premises; the equations treated in the two papers are essentially the same. *S. Chandrasekhar.*

Busbridge, Ida W. On emission lines in stellar spectra. Monthly Not. Roy. Astr. Soc. **112**, 45–54 (1952).

The problem of the formation of absorption lines in stellar atmospheres in the framework of the so-called Milne-Eddington model has been exactly solved [S. Chandrasekhar, Radiative transfer, Oxford, 1950, p. 84; these Rev. **13**, 136] and the theory gives an expression for the emergent flux in terms of the moments of the H-functions defined by

$$H(\mu)=1+\tfrac{1}{2}(1-\lambda)\mu H(\mu)\int_0^1\frac{H(\mu')}{\mu+\mu'}d\mu'$$

(here $\lambda<1$ is a constant) and certain other parameters depending on the ratio of the line to the continuous absorption coefficients. It has been known for some time [cf. Underhill, Astrophys. J. 110, 340–354 (1949); these Rev. 11, 409] that for certain values of these latter parameters the line will appear in emission instead of in absorption. The precise conditions for this to happen are examined in this paper. The analysis requires a knowledge of the behavior of the moments of the H-function for $\lambda\to 1$ and $\lambda\to 0$ and these behaviors are determined. *S. Chandrasekhar.*

RELATIVITY

***Einstein, Albert. The meaning of relativity. Third edition, revised, including The Generalized Theory of Gravitation.** Princeton University Press, Princeton, N. J., 1950. iv+165 pp. $2.50.

The present edition of this book, the revised third edition, differs from the third edition in that Appendix II entitled "Generalized Theory of Gravitation" has been rewritten. This review will be wholly concerned with this material [for a review of the 2nd edition see these Rev. 7, 87]. In this version Einstein discusses but does not solve the problem of characterizing the manifold of solutions of the field equations that he considers the natural generalization of the field equations of general relativity.

This generalization is based on the existence of a "total field comprising the entire physical reality" which is described by a non-symmetrical tensor field g_{ik}. A non-symmetrical affine connection Γ^i_{kl} is introduced and determined by the g_{ik} via the equations

$$(1)\qquad g_{ik,l}=g_{mk}\Gamma^m_{il}+g_{im}\Gamma^m_{lk},$$

where the comma denotes the ordinary derivative. The generalized field equations are then (1) and the statement that all contracted curvature tensors derived from the Γ^i_{kl} vanish. This condition may be written as

$$(2)\qquad R_{kl}=\Gamma^i_{kl,i}-\Gamma^i_{kl}\Gamma^j_{ij}-\tfrac{1}{2}[\Gamma^i_{ki,l}+\Gamma^i_{il,k}]+\Gamma^i_{kl}\Gamma^j_{ji}=0$$

and (3) $\Gamma_i=0$, where $\Gamma_i=\Gamma^j_{ij}$ and the symbol $-$ or $\sim$ under a pair of indices of a quantity denotes the symmetric or antisymmetric part, respectively, of that quantity with respect to these indices.

The equations (1), (2), and (3) are referred to as equations I. If in this system equations (1) determine Γ^i_{kl} uniquely as functions of the g_{ik} and their derivatives, then equations (2) and (3) are a system of twenty differential equations for the sixteen quantities g_{ik}. The author states, "It is, however, not at all obvious that the manifold of solutions of system (I) is extensive enough to satisfy the requirements of a physical theory ('compatibility'). . . . In spite of considerable efforts I did not succeed in proving the compatibility of system I. At the present time the possibility cannot be excluded that the system I, judged as a physical theory, does not have enough solutions although from a formal mathematical point of view it is the only satisfactory one".

The formal requirements that are referred to above are the existence of the non-symmetric g_{ik} and the principle of "Hermitian symmetry": The equations are such that if the g_{ik} and Γ^i_{kl} are replaced by $\bar g_{ik}=g_{ki}$ and $\bar\Gamma^i_{kl}=\Gamma^i_{lk}$, and the indices i and k are interchanged in the resulting expressions, the equations remain unaltered. The author discusses the compatibility of systems of equations weaker than system I, that is, which are implied by I but do not imply I, which are such that they are derivable from variational principles. System I is not so derivable. This discussion is very illuminating as an interpretation of the Bianchi identities. However, in spite of it there is still the open mathematical question concerning the manifold of solutions of system I. The question as to the nature of the Γ^i_{kl} as functions of the g_{ik} and their derivatives is not discussed although these equations play an important role in system I.

The author's beautifully simple and lucid style is used in Appendix II as well as in earlier portions of the book. As a result this book remains one that can be highly recommended to specialists and to the general reader.

A. H. Taub (Urbana, Ill.).

Infeld, L. The new Einstein theory and the equations of motion. Acta Phys. Polonica 10, 284–293 (1951).

The author determines approximately the equations of motion for a particle in Einstein's Generalized Field Theory from the field equations of that theory by using the methods that Einstein and he had developed previously [Canadian J. Math. 1, 209–241 (1949); these Rev. 11, 59]. He obtains the result that, up to fourth order, the equations of motion are the same as in general relativity and are uninfluenced by the "electromagnetic field". *A. H. Taub* (Urbana, Ill.).

Castoldi, Luigi. Attorno a una recente teoria relativistica. Atti Accad. Ligure 7, 307–310 (1951).

The author presents formal objections to the definition of covariant derivative, with respect to an unsymmetrical affine connection, adopted by Einstein in his recent unified field theory. *H. P. Robertson* (Pasadena, Calif.).

Ikeda, M. On the approximate solutions of the unified field theory of Einstein and Schrödinger. Progress Theoret. Physics 7, 127–128 (1952).

In this paper the linearization of the unified field equations is obtained in a slightly different manner than previously.

M. Wyman (Edmonton, Alta.).

Haywood, J. H. The equations of motion and coordinate condition in general relativity. Proc. Phys. Soc. Sect. A. 65, 170–175 (1952).

The author continues the work of Einstein, Papapetrou, and others on the effect of the coordinate condition on the equations of motion in General Relativity. In the present paper a new coordinate condition is investigated. Although new second order terms appear in the equations of motion for the two-body problem, it is found that they do not affect the secular motion of the center of mass or the rotation of periastron of the two-body system. *M. Wyman.*

Gel'man, E. E. Real spinors in the general theory of relativity. Leningrad. Gos. Univ. Učenye Zapiski 120, Ser. Fiz. Nauk 7, 79–133 (1949). (Russian)

The paper develops the mathematics of real, four-component spinors and their relations to vectors and tensors. The transformations of these spinors are considered under spin transformations and under gradient, or gauge, transformations. The spinors are divided into two classes according to the way in which they transform under the latter. The usual complex spinors are shown to be linear combinations of real spinors of both classes. Covariant derivatives of spinors are defined, taking into account the possibility of gauge transformations. The wave equation for a particle is

set up in terms of real spinors and is found to contain one spinor of each class. The equation can be put into the form of the Dirac equation, involving two-component spinors, as given by Infeld and van der Waerden [S.-B. Preuss. Akad. Wiss. 1933, 380–401], or into the form involving a four-component complex spinor, as given by Fock [Z. Physik 57, 261–277 (1929)]. An expression for the energy-density tensor in terms of real spinors is derived.

N. Rosen (Chapel Hill, N. C.).

Maravall Casesnoves, Darío. The structure of media with axial symmetry in general relativity. Application to galaxies. Revista Mat. Hisp.-Amer. (4) **11**, 277–287 (1951). (Spanish)

L'A. étudie, pour les équations d'Einstein avec constante cosmologique, les solutions statiques de la forme

$$ds^2 = e^{\nu}dt^2 - (e^{\lambda}dr^2 + r^2d\phi^2 + dz^2)$$

où λ et μ sont fonctions de r. Trois cas sont envisagés pour le tenseur d'impulsion-énergie: celui du fluide parfait, celui des particules indépendantes (galaxies), celui du vide. Dans chaque cas, l'A. donne les équations détérminant λ et μ et forme en première approximation (r petit) les équations différentielles du mouvement d'une particule. On notera que dans l'hypothèse du fluide parfait, la densité admet une distribution maxwellienne et, en première approximation, la vitesse angulaire des molécules est constante. Dans le second cas, qui correspond aux galaxies, la densité stellaire est en première approximation inversement proportionnelle à r et la vitesse angulaire à $r^{1/2}$. La régularité globale des solutions n'est pas envisagée. *A. Lichnerowicz* (Paris).

Frankl', F. I. On gravitational waves and on the motion of gases in strong varying gravitational fields. Doklady Akad. Nauk SSSR (N.S.) **84**, 51–54 (1952). (Russian)

On the basis of the gravitational equations of Einstein the problem of determining the gravitational field and the motion of gases at given initial conditions on a space-like hypersurface is considered. This problem has a unique solution. An arbitrary transformation of the coordinate system is allowed as long as it does not change the initial conditions.

L. Infeld (Warsaw).

Riesz, Marcel. Sur le potentiel de Liénard-Wiechert attaché à une ligne d'univers. C. R. Acad. Sci. Paris **234**, 2159–2161 (1952).

The author has shown previously [Acta Math. **81**, 1–223 (1949); these Rev. **10**, 713] using an explicit calculation that the Liénard-Wiechert potential due to a charge moving along a world line L satisfies the partial differential equation

$$A(x) = \frac{1}{8\pi}\Box_x Z_{ret}$$

where $\Box_x (= g^{ij}\partial^2/\partial x^i \partial x^j)$ is the D'Alembertian operator and Z_{ret} are the coordinates of the point of intersection of the world line L with the backward branch of the light core emanating from the point x of space time. In this note a simple direct proof of this result is given using some results obtained in the paper cited above. *A. H. Taub.*

Riesz, Marcel. Sur le potentiel retardé attaché à un courant continu. C. R. Acad. Sci. Paris **234**, 2260–2261 (1952).

The author generalizes the result in the paper reviewed above, concerning the vector potential due to a point charge, to the case of a continuous current vector which satisfies the continuity equation. *A. H. Taub* (Urbana, Ill.).

†v. Laue, M. Die Relativitätstheorie. Erster Band. Die spezielle Relativitätstheorie. 5th ed. Friedr. Vieweg & Sohn, Braunschweig, 1952. viii+198 pp. (1 plate).

In this fifth edition of his classical book on the special theory of relativity the author has added some new material and revised other portions of the book. As in earlier editions the main aim of this edition is to show the application of the theory of special relativity to various problems and experiments in classical physics. Chapter I is devoted mainly to a discussion of the experiments which show the contradiction between the Newtonian system of relativity and electro-magnetic phenomenon. The Lorentz transformations are then introduced and applied to the experiments previously discussed. The Minkowski formalism for picturing space-time as a four-dimensional space with an indefinite quadratic form is then given. The form is made definite by the introduction of a purely imaginary time-coordinate. Chapter II is devoted to a discussion of tensor algebra in the Minkowski space. Chapter III is concerned with the four-dimensional formulation of Maxwell's electromagnetic theory in free space. Chapter IV discusses critically Minkowski's theory of the electrodynamics of moving bodies. Chapter V gives a brief introduction of the application of special relativity to particle mechanics, hydrodynamics, and thermodynamics. This discussion is much shorter than that presented in earlier versions of the book.

The book fulfills its main aim referred to above admirably. However the modern student of physics and mathematics is very often interested in the theory of special relativity for reasons which do not involve classical physics and unfortunately he will have to go elsewhere to satisfy these interests. Nevertheless for someone concerned with classical physics as viewed from the theory of special relativity this well written and well organized book is highly recommended.

A. H. Taub (Urbana, Ill.).

Macke, W. Begründung der Speziellen Relativitätstheorie aus der Hamiltonschen Mechanik. Z. Naturforschung **7a**, 76–78 (1952).

The author generalizes the Hamiltonian function for a dynamical system so that energy and time are canonically conjugate variables. He then shows that the transformation between two inertial systems is canonical if it is a Lorentz transformation. *A. H. Taub* (Urbana, Ill.).

Pignedoli, Antonio. Sulla dinamica relativistica del punto materiale. Accad. Sci. Modena. Atti Mem. (5) **9**, 256–273 (1951).

The author integrates the special relativistic equations of motion of a particle subjected to various external fields. These include the cases of rectilinear motion under a force depending only upon position or upon velocity, motion under a central force including the Newtonian force, and the motion of a charged particle in the field of a rectilinear current. *H. P. Robertson* (Pasadena, Calif.).

Galli, Mario. Le deformazioni relativistiche di un cilindro rotante. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 86–92 (1952).

A cylinder of radius r rotates about its axis with constant angular velocity ω and at the same time moves along that axis with constant velocity v. Each point on the surface of the cylinder has a velocity V, the resultant of components

$r\omega$ and v. An element ds of the surface of the cylinder is changed by the motion in accordance with the rule that the component of ds parallel to V undergoes a FitzGerald-Lorentz contraction and the component perpendicular to V remains unchanged. Proceeding in this way, the author confirms formulae for the deformation of the cylinder obtained by H. Ives [J. Opt. Soc. Amer. **29**, 472–478 (1939)], and discusses why these formulae differ from those of A. Eagle [Philos. Mag. (7) **28**, 592–595 (1939)]. He remarks that the problem of the rotating cylinder has been much discussed and a variety of results obtained; he does not regard the present results as wholly satisfactory. [To the reviewer, the problem appears to be that of prescribing the congruence of world lines of the particles of the moving cylinder, correlating each world line to one of the straight world lines of the cylinder before it is set in motion. There is no mathematical problem here until further conditions are stated (conditions of rigidity). If the condition is that the normal Minkowskian distance between adjacent world lines is unchanged (equivalently, that the FitzGerald-Lorentz contraction holds for all elements), then the problem of the rotating cylinder has no solution [cf. Herglotz, Ann. Physik (4) **31**, 393–415 (1910)]. In the present paper this condition is applied, but only to the world lines of particles on the surface of the cylinder.] *J. L. Synge* (Dublin).

Synge, J. L. Orbits and rays in the gravitational field of a finite sphere according to the theory of A. N. Whitehead. Proc. Roy. Soc. London. Ser. A. **211**, 303–319 (1952).

This paper contains a philosophical introduction the object of which is to place Whitehead's theory (simplified and generalized by the author) in a philosophical perspective. Whitehead's theory is generalized for the case of continuous masses. The gravitational field inside and outside of a sphere with spherical density is calculated. The motion of test particles in such a field is derived from a variational principle. The deflection of light rays is the same as in Einstein's theory, the perihelion motion is "practically" the same. Also "the same red-shift as Einstein's is given by an obvious interpretation of Whitehead's theory". Kepler's third law appears in a slightly modified form. *L. Infeld.*

Zagar, Francesco. Sulla espansione dell'Universo. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 12–16 (1952).

The author sets up a Newtonian model of the expanding universe, and then objects to it on the ground that in it the velocities of the galaxies must at some time in the past have exceeded that of light. He also emphasizes the point that, as in the general relativistic theories, the time scale for the universe is exceedingly short. Objections are raised to the recent hypothesis of the continuous creation of matter.
H. P. Robertson (Pasadena, Calif.).

Zagar, Francesco. Sulla espansione dell'Universo. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 239–245 (1952).

The author assumes an inverse square repulsive force between nebulae, and deduces a velocity of recession proportional to their distance apart. Radiation pressure, offsetting gravity in the earlier stages of the expansion, is advanced as such a force, augmented perhaps by the "absorption of gravity" in traversing space.
H. P. Robertson (Pasadena, Calif.).

MECHANICS

Bereis, R. Die Fernpolstellung der ebenen Bewegung. Österreich. Ing.-Arch. **6**, 246–255 (1952).

An earlier paper [same journal **5**, 246–266 (1951); these Rev. **13**, 292] on the study of plane motions by means of the complex variable is extended to consider the special cases in which the instantaneous center of rotation is moved off to infinity. The inflection circle becomes a line; the envelope of the affine normals at the points along a radial line is a parabola; the locus of umbilical points is a rectangular hyperbola. Applications are made to the motions of a centered crosshead mechanism and of a sliding crosshead mechanism. *M. Goldberg* (Washington, D. C.).

Buckens, F. Sur une propriété de similitude des configurations moyennes de mobiles indéformables. Ann. Soc. Sci. Bruxelles. Sér. I. **66**, 61–69 (1952).

As a plane moves over a fixed plane, each point in the moving plane describes a curve in the fixed plane. If $p(t)$, an arbitrary function of the time t, is used as a weighting function, then each curve can be replaced by a point which is the weighted mean position of the moving point. By the use of the complex variable, it is shown that the configuration made by a set of weighted means corresponding to a selected set of points is similar to the configuration of the selected set of points. In particular, if $p(t)=1/t$, then the configurations have the same size. If the projections of the motions are expanded as a Fourier series, the Fourier coefficients correspond to the mean positions. A motion may then be represented as the sum of rotating vectors in which the rates of rotation are proportional to the orders of the coefficients [cf. Biezeno and Grammel, Technische Dynamik, Springer, Berlin, 1939, p. 847]. The method is applied to the motions of a slider and crank mechanism (a crosshead mechanism). The uses of various weighting functions are discussed. *M. Goldberg* (Washington, D. C.).

Gale, David. An indeterminate problem in classical mechanics. Amer. Math. Monthly **59**, 291–295 (1952).

Let A, B, C be three balls which move on a straight line and collide with one another. The author shows that the result of the collisions may be entirely different according as the first contact is between A and B or between B and C; if the contacts are simultaneous, the outcome is mathematically indeterminate. He suggests that this indeterminacy may exist in nature, in the sense that the repetition of the same experiment may lead to different results. In connection with the present problem, this view does not appeal to the reviewer, because the indeterminacy disappears if we think of finite spheres in terms of the theory of elasticity, or even if we analyse the impacts into periods of compression and restitution [cf. E. J. Routh, Elementary rigid dynamics, 6th ed., Macmillan, London, 1897, p. 150; J. L. Synge and B. A. Griffith, Principles of mechanics, 2nd ed., McGraw-Hill, New York, 1949, p. 233; for a review of the 1st ed. see these Rev. **3**, 213]. *J. L. Synge.*

Marsicano, Fénix R. On the stability of movement of a nonhomogeneous disc which rests on a smooth horizontal plane. Ciencia y Técnica **119**, no. 601, 9–15 (1952). (Spanish. French summary)

The author considers a circular non-homogeneous disk which moves with its edge in contact with a fixed smooth

horizontal plane. It is shown that a motion in which the centroid of the disk has its highest possible position is stable, and that another motion in which the centroid has its lowest possible position is unstable. *L. A. MacColl.*

Synge, J. L. On a case of instability produced by rotation. Philos. Mag. (7) **43**, 724–728 (1952).

The author discusses the rather unexpected properties of a recently devised toy top, and shows that these properties are simple consequences of Routh's theory of gyroscopic stability. *L. A. MacColl* (New York, N. Y.).

Stoppelli, Francesco. Un'osservazione sull'applicabilità del principio dell'effetto giroscopico ai sistemi di solidi mutuamente vincolati. Ricerche Mat. **1**, 20–26 (1952).

The author discusses certain approximations in the theory of a dynamical system consisting of two solids, S_1 and S_2, where S_2 is rotating rapidly about an axis which is fixed in S_2 and passes through the centroid of S_1. It is shown that the legitimate approximations are not necessarily identical with those which would be made if S_2 were a free solid subjected to the actual applied forces and to a force equal to the reaction of S_1. *L. A. MacColl* (New York, N. Y.).

Bödewadt, U. T. Der symmetrische Kreisel bei zeitfester Drehkraft. Math. Z. **55**, 310–320 (1952).

The author solves the differential equations of motion of a symmetrical top subject to forces such that the torques about the principal axes of inertia of the top are constant with respect to time. It is suggested that the modern use of jet propulsion now gives such problems an interest which they have previously lacked. *L. A. MacColl.*

Aymerich, Giuseppe. Trasformazioni di Appell nel caso di forze lineari nelle velocità. Rend. Sem. Fac. Sci. Univ. Cagliari **20** (1950), 184–192 (1951).

It is shown that a system of equations of the form

$$d^2x_i/dt^2=\sum_{j=1}^{n}\varphi_{ij}(x_1,\cdots,x_n)(dx_j/dt)+f_i(x_1,\cdots,x_n),\qquad i=1,2,\cdots,n,$$

is left invariant by a transformation of the form

$$y_i=y_i(x_1,\cdots,x_n),\quad dt=G(x_1,\cdots,x_n)dT,$$

provided that certain conditions are satisfied. The theorem is a generalization of a theorem of Appell in which the φ's are all zero [cf. Amer. J. Math. **12**, 103–114 (1890)]. Some further results of the same type are obtained in case the equations are of Lagrangian form. *D. C. Lewis.*

Nadile, Antonio. Estensione del teorema di Liouville sull'integrazione dei sistemi canonici. Atti Sem. Mat. Fis. Univ. Modena **4**, 45–53 (1950).

The theorem that a system of canonical equations of order $2n$ can be completely integrated by quadratures, if n particular independent integrals in involution are given, is given its appropriate generalization to the case when the differential equations are no longer of the usual Hamiltonian type. The equations are those for a holonomic conservative dynamical system in which the momenta are replaced by linear combinations of the momenta as one of the sets of n variables. The coefficients in these linear combinations are arbitrary functions of the coordinates and the time.

D. C. Lewis (Baltimore, Md.).

Nadile, Antonio. Estensione del teorema di Liouville sull'integrazione dei sistemi canonici ai sistemi materiali anolonomi. Atti Sem. Mat. Fis. Univ. Modena **4**, 54–63 (1950).

The author generalizes the work of the paper reviewed above to the case of non-holonomic equations. In this connection, it is to be remembered that the theorem of Liouville cannot be expected to hold for non-holonomic systems unless certain equalities involving the constraint coefficients are known to hold. *D. C. Lewis* (Baltimore, Md.).

Mendes, Marcel. Équations de Lagrange et équations canoniques. C. R. Acad. Sci. Paris **233**, 1574–1575 (1951).

Étude du passage des équations de Lagrange aux équations canoniques: réciprocité des fonctions génératrices des deux systèmes d'équations. Équivalence des systèmes canoniques à même nombre de degrés de liberté. Généralisation des équations canoniques dans un cas étudié par Delassus. (Author's summary.) *D. C. Lewis* (Baltimore, Md.).

Meffroy, Jean. Sur une généralisation des équations canoniques. Bull. Astr. (2) **16**, 213–219 (1952).

Systems of ordinary differential equations of the form

$$\frac{dx_i}{dt}=\frac{\partial F}{\partial y_i}+\frac{\partial F}{\partial z_i}+\frac{\partial F}{\partial u_i},$$

$$\frac{dy_i}{dt}=-\frac{\partial F}{\partial x_i}+\frac{\partial F}{\partial z_i}+\frac{\partial F}{\partial u_i},$$

$$\frac{dz_i}{dt}=-\frac{\partial F}{\partial x_i}-\frac{\partial F}{\partial y_i}+\frac{\partial F}{\partial u_i},$$

$$\frac{du_i}{dt}=-\frac{\partial F}{\partial x_i}-\frac{\partial F}{\partial y_i}-\frac{\partial F}{\partial z_i},\qquad i=1,\cdots,n,$$

where F is a given function of the four groups of variables, $x_1,\cdots,x_n, y_1,\cdots,y_n, z_1,\cdots,z_n, u_1,\cdots,u_n$, are considered as generalizations, in the case of four groups of variables, of the usual Hamiltonian canonical equations, in the case of two groups. Necessary and sufficient conditions that a transformation on the variables should preserve the above form are investigated. The methods can be generalized to the case when F is a function of any even number of groups of variables, and they lead, of course, to certain generalizations of the Poisson brackets. *D. C. Lewis.*

Mineur, Henri. Sur les points singuliers des systèmes canoniques admettant un nombre d'intégrales premières uniformes en involution égal à celui des degrés de liberté. C. R. Acad. Sci. Paris **234**, 1844–1846 (1952).

Canonical systems of n degrees of freedom and admitting n independent first integrals in involution are considered. A point M of the phase space is said to be singular of order p if the n by $2n$ functional matrix of the n first integrals with respect to the $2n$ phase variables has its rank lowered to $n-p$ at M. It is shown that the singular points of order 1 occur on a finite number of varieties of $2n-2$ dimensions.

D. C. Lewis (Baltimore, Md.).

Šul'gin, M. F. A theorem on properties of integrals of the dynamical equations of S. A. Čaplygin. Doklady Akad. Nauk SSSR (N.S.) **84**, 899–902 (1952). (Russian)

Continuing the work of two earlier papers [same Doklady **75**, 349–351 (1950); **81**, 23–26 (1951); these Rev. **12**, 414; **13**, 594], the author establishes a theorem, analogous to

the classical theorem of Poisson, on integrals of Chaplygin's dynamical equations of a nonholonomic conservative system [Collected works, vol. I, Moscow-Leningrad, 1948, pp. 57–75]. The theorem is illustrated by an example of motion of a rigid body parallel to a plane [cf. Chaplygin, loc. cit., pp. 21–24]. *E. Leimanis* (Vancouver, B. C.).

Madhava Rao, B. S. Notes on dynamics. II. Characterisation of dynamical systems. Half-Yearly J. Mysore Univ. Sect. B., N.S. **5**, 35–43 (1944).

It is well known that the Hamiltonian canonical equations of dynamics admit certain absolutely invariant integrals of dimension $2k$, where $k \leqq n$, the number of degrees of freedom. In this paper the author shows that, conversely, if a system of differential equations of order $2n$ admits one of these invariant integrals for which $k<n$, then the system is canonical. The method fails for $k=n$, but the author presents an alternative method indicating that the theorem is true even for $k=n$. The fact that the author is wrong in this case is indicated by the following trivial example of a non-canonical system $dp_1/dt=-3p_1$, $dp_2/dt=p_2$, $dq_1/dt=q_1$, $dq_2/dt=q_2$, which admits the invariant integral $\iiiint dp_1dp_2dq_1dq_2$. The author also proves various other converses of well known theorems giving certain characteristic properties of Hamiltonian systems, having to do with Poisson brackets and the group properties of first integrals. *D. C. Lewis* (Baltimore, Md.).

Mihailovitch, D. Interprétation de quelques résultats dans un cas spécial du problème des trois corps. Bull. Soc. Math. Phys. Serbie **2**, nos. 3–4, 27–34 (1950). (Serbo-Croatian. French summary)

The paper deals with an approximate solution which H. G. Block [Dissertation, Lund, 1909] had found to be valid for a reasonably short time interval after the instant t_0 of the collision of the three bodies. The approximation is

$$\frac{d\overrightarrow{R_i}}{dt}=\vec{a}_i\frac{d}{dt}(t-t_0)^{2/3} \quad (i=1, 2, 3),$$

where the R_i are distances from the center of inertia of the system, and a_i are constants. The author gives a kinematic interpretation of this approximation and shows that for each center of mass it is equivalent to replacing the instantaneous relative motions of the barycenter and of the tangent on the trajectory by a system of straight lines parallel to these motions. There are thus three systems of parallel lines and each line has two corresponding lines in the other two systems. Each group of corresponding lines has a common point of intersection which lies in the plane of motion of the barycenters and the locus of each point of intersection is a curve of the same type as that of the trajectories of the barycenters themselves. *L. Jacchia.*

Michailovitch, Dobrivoje. Contribution à l'étude d'un problème particulier de n corps. Bull. Soc. Math. Phys. Serbie **3**, no. 1–2, 11–34 (1951). (Serbo-Croatian. French summary)

The author generalizes some results previously obtained by him in the problem of the collision of three bodies, using H. G. Block's [Dissertation, Lund, 1909] approximations. The theorem on the gravitational pole is extended to the case of n bodies and subsequently applied to the special case of 4 bodies in which the particular solutions correspond to a configuration in the shape of a regular tetrahedron. The author finds that in the problem of n bodies the projections of the individual vectors $\mathbf{r}_i$ on the three axes of an orthogonal system are linearily correlated, so that the number of equations can be reduced. In the case of the tetrahedron six equations suffice instead of nine. Similar linear correlations are found for the projections of the gradient vector of the force function. *L. Jacchia* (Cambridge, Mass.).

***Popoff, Kyrille. Le mouvement d'un projectile autour de son centre de gravité.** Mémor. Sci. Math., no. 117. Gauthier-Villars, Paris, 1951. 51 pp. 550 francs.

This booklet provides a unified exposition of the theories and results up to 1939 concerning the motion of a projectile about its center of gravity; the author himself has taken an active part in this development. The paper concludes with an extensive bibliography; and, in general, it gives a rather clear summary of the situation in this field. *E. Leimanis* (Vancouver, B. C.).

Sexl, Theodor. Über die Bewegung eines Massenpunktes in einem widerstehenden Mittel veränderlicher Dichte. Acta Physica Austriaca **5**, 148–151 (1951).

Consider the motion of a material point in a resisting medium whose density decays exponentially with altitude. Assume that the positive z-axis is directed downward and that the resistance to the motion of the point is proportional to the density of the medium and to the nth power of the instantaneous velocity $\dot{z}$ of the moving point. The equation of motion is then $\ddot{z}=g-k\exp(\alpha z)\dot{z}^n$. It is integrated in the closed form for $n=1$ and $n=2$ with the initial conditions $z=H$, $\dot{z}=0$ for $t=0$. *E. Leimanis* (Vancouver, B. C.).

Bolz, Ray E. Dynamic stability of a missile in rolling flight. J. Aeronaut. Sci. **19**, 395–403 (1952).

The equations of motion of a symmetric spinning missile are obtained. The missile may be jet or rocket-propelled or coasting under accelerating or decelerating conditions. In addition to the usual set of aerodynamic forces and couples, the effects of misaligned surfaces, canted surfaces, jet malalignment, and linear accelerations in the plane normal to the missile axis are considered; gravity is neglected. An unusual feature of the paper, which in other respects follows lines similar to those laid down by Fowler et al. [Philos. Trans. Roy. Soc. London. Ser. A. **221**, 295–387 (1920)] and Nielsen and Synge [Quart. Appl. Math. **4**, 201–226 (1946); these Rev. **8**, 100] is that the equations are referred to axes fixed in the missile. Stability conditions similar to those obtained by Nielsen and Synge, Kelley, McShane and Reno (unpublished), and the reviewer [Philos. Trans. Roy. Soc. London. Ser. A. **241**, 457–585 (1949); these Rev. **10**, 749] are obtained and discussed. In an appendix a simple thrust equation for a ram-jet is derived. *R. A. Rankin.*

Hydrodynamics, Aerodynamics, Acoustics

Castoldi, Luigi. Linee sostanziali nel moto di un continuo deformabile e moti con linee di flusso (e di corrente) "sostanzialmente permanenti." Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **14(83)**, 259–264 (1950).

The author derives Zorawski's criterion for permanent vector lines [Anz. Akad. Wiss. Krakau **1900**, 335–342], which he attributes to Lampariello [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **26**, 383–387 (1937)]. He notes some easy consequences, e.g. that the

stream-lines are material lines if and only if they are steady. [A derivation without use of infinitesimals was given by R. Prim and the reviewer, Proc. Amer. Math. Soc. 1, 32–34 (1950); these Rev. 11, 696.] *C. Truesdell.*

Garabedian, P. R., and Spencer, D. C. Extremal methods in cavitational flow. J. Rational Mech. Anal. 1, 359–409 (1952).

On sait que les méthodes variationnelles ont été utilisées avec succès pour étudier le problème de l'unicité dans la théorie des écoulements irrotationnels avec lignes à pression constante. Schiffer a montré récemment comment ces méthodes variationnelles permettaient également d'obtenir des théorèmes d'existence. En reprenant les idées de Schiffer, les auteurs développent une théorie variationnelle de ces écoulements.

La première partie du travail est consacrée à l'établissement des deux principes qui sont à la base de la théorie: le principe du minimum de la masse virtuelle et le principe du maximum de la moindre vitesse. Le premier principe peut s'énoncer sous la forme suivante: soit un mouvement irrotationnel, avec une vitesse uniforme à l'infini égale à un, autour d'un objet B, de volume A, dont la frontière C contient des arcs L non fixés, et soit $\phi=x+ax/r^3+\cdots$, $r^2=x^2+y^2+z^2$, le développement autour du point à l'infini du potentiel des vitesses; lorsque les arcs variables L sont tels que l'expression $2\pi a-\lambda A$ est minimum, la vitesse sur ces arcs L est alors constante et égale à λ. Des extensions de ce principe sont données pour le cas où il y a le champ de la pesanteur et pour le cas d'un mouvement plan avec une circulation autour de B donnée. Le principe du minimum de la moindre vitesse est le suivant: étant donné un écoulement de révolution par rapport à Ox, autour d'un corps B, dont la demie méridienne C, symétrique par rapport à Oy, contient un arc L inconnu ayant la même symétrie; lorsque l'arc L est tel que le minimum de la vitesse sur cet arc est maximum, alors la vitesse sur L est constante.

Ces principes variationnels sont ensuite utilisés pour démontrer l'existence de certains schémas hydrodynamiques classiques tels que: cavitation plane entre deux plans perpendiculaires au courant, sillage plan à distance finie derrière un obstacle avec une pression supérieure à la pression à l'infini. Pour les mouvements de révolution les principes établis ne donnent qu'une approximation polygonale des lignes libres. Dans le dernier paragraphe est analysé le problème d'une cavitation non permanente, posé par l'entrée d'un projectile dans l'eau.

Les auteurs ont volontairement limité à ces quelques exemples l'application de la méthode variationnelle, mais on conçoit que cette méthode soit extrêmement générale pour obtenir des théorèmes d'existence dans les problèmes de mouvements avec lignes à pression constante.

R. Gerber (Grenoble).

Garabedian, P. R., and Royden, H. L. A remark on cavitational flow. Proc. Nat. Acad. Sci. U. S. A. 38, 57–61 (1952).

Deux démonstrations de l'existence des cavitations planes ont été données récemment par Garabedian et Spencer [voir l'analyse ci-dessus] et par Schiffer; ces démonstrations qui sont indépendantes reposent sur une propriété extrêmale des lignes d'égale pression qui avait été déjà utilisée par Schiffer. Les auteurs de la présente note donnent une nouvelle démonstration élémentaire, basée sur la même propriété extrêmale, qui doit se prêter à une généralisation aux mouvements avec symétrie de révolution. Les courbes extrêmales du problème variationnel sont approchées par des polygones extrêmaux ce qui permet de retrouver très simplement la propriété de ces courbes d'être des lignes d'égale pression. *J. Kravtchenko* et *R. Gerber.*

Lavrentieff, M. Sur la théorie exacte des ondes longues. Akad. Nauk Ukrain. RSR. Zbirnik Prac' Inst. Mat. 1946, no. 8, 13–69 (1947). (Ukrainian. Russian and French summaries)

Consider two-dimensional steady irrotational motion of a heavy incompressible ideal fluid in a channel. Let $y=y_0(x)$ be the equation of the bottom of the channel and $y=y(x)$ the equation of the free surface. Let h denote the rate of discharge of the fluid per unit width. Let $f(z)=\varphi+i\psi=\zeta$ be the complex potential mapping the strip $y_0(x)\leqq y\leqq y(x)$ in the z-plane onto the strip $0\leqq\psi\leqq h$ in the ζ-plane. A fundamental problem in the theory of surface waves is to find the class of curves $y(x)$ such that

$$|f'(z)|^2+2gy-B=0 \quad \text{for} \quad y=y(x),$$

where g is the acceleration of gravity and B is some constant. Levi-Cività and A. I. Nekrasov independently proved the existence of periodic solutions to this problem under certain restrictions. In this paper the author establishes the existences of periodic solutions over a periodic bottom for sufficiently large period, and, in addition, obtains as a limiting case the existence of a solitary wave over a flat bottom. The proof, developed in a long series of lemmas, employs the variational methods in conformal mapping developed by the author and depends in particular on his earlier paper on jets [Mat. Sbornik N.S. 4(46), 391–458 (1938)].

Misprints are so numerous and frequently of such a confusing nature that trying to follow the details of many of the proofs is (to the reviewer) a frustrating experience. In addition, the equations are not always dimensionally consistent so that the hydrodynamic significance of some of the restrictions is not always clear; however, the preliminary announcement of these results [C. R. (Doklady) Acad. Sci. URSS (N.S.) 41, 275–277 (1943); these Rev. 6, 191] provides some of the necessary background. The importance of the results would seem to have merited more careful exposition and printing (as well as publication in a more widely read journal and language). *J. V. Wehausen.*

Birkhoff, Garrett. A new theory of vortex streets. Proc. Nat. Acad. Sci. U. S. A. 38, 409–410 (1952).

Remarks are made about vortex streets behind obstacles; they are too condensed to be clearly intelligible but will, it is stated, be expanded elsewhere. *M. J. Lighthill.*

Imai, Isao, Kaji, Ikuo, and Umeda, Kwai. Mapping functions of the N.A.C.A. airfoils into the unit circle. J. Fac. Sci. Hokkaido Univ. Ser. II. 3, 265–304 (1950).

A method of conformal mapping developed by one of the authors [I. Imai, J. Soc. Aeronaut. Sci. Japan 9, 865 (1942); I. Imai and K. Sato, J. Aeronaut. Res. Inst. Tokyo Imp. Univ. 247, 91 (1945)] is sketched and a numerical procedure based on this method is described. The method depends upon the reduction of the problem of calculating the function which maps a simply connected region conformally on the exterior of a circle to the problem of solving a singular integral equation of the Theodorsen-Garrick type. This method is applied to 32 N.A.C.A. airfoils to calculate several terms of the series which represents the mapping function and some empirical results are given relating these coeffi-

cients and the parameters of the airfoils. Extensive tables are given. *C. Saltzer* (Cleveland, Ohio).

van Heemert, A. Theory of the calculation of load-distribution for yawed and swept wings in incompressible flow. Nationaal Luchtvaartlaboratorium, Amsterdam. Report **F. 91**, 32 pp.+iii (1951).

This is an attempt to handle the equations of lifting-surface theory, in the incompressible-flow, thin-wing case, by forming successive moments of the chordwise circulation distribution. By means of approximations, which appear to limit the theory to wings of high aspect ratio, a system of integro-differential equations is obtained for these moments in terms of certain chordwise integrals of the downwash. A method of solution is proposed. The theory is analogous to one proposed by Reissner [Proc. Nat. Acad. Sci. U. S. A. **35**, 208–215 (1949); these Rev. **10**, 753], but the author points out that the results are not the same.

W. R. Sears (Ithaca, N. Y.).

van Heemert, A. The calculation of the downwash at the surface of a lifting plane in steady flow. Tables, graphs and some simple checks. Part 1. Theory. Part 2. Tables and graphs. Nationaal Luchtvaartlaboratorium, Amsterdam. Report **F. 58**, i+40 pp. (14 tables+16 graphs) (1950).

In two previous reports [same Reports **F. 51**, **F. 55** (1949); these Rev. **13**, 175, 164] the author has developed a method for the calculation of the downwash at a lifting surface with arbitrary but sufficiently regular plan form. In the present report he first presents a summary of his method which essentially is the introduction of Fourier expansions (with additional terms accounting for the singularity at the leading edge) of the vorticity distribution in suitably chosen coordinates into the integral equation of the lifting surface. The region of integration is split up into two parts: a) a strip surrounding the pivotal point and bordered by two parallel lines extending in the direction of the wake from the leading edge to infinity, and b) the remaining parts of the surface and the wake. In performing the integration over a) some simplifying assumptions are made.

Tables and charts are then given for these integrals. As a partial check on the applicability of the method the downwash is calculated in some simple cases where exact solutions are possible and good agreement is obtained. Finally a few remarks follow on the "reverse problem" where the downwash is given over the wing area and the vorticity distribution has to be determined. *H. Parkus.*

van Heemert, A. A generalisation of Prandtl's equation. Nationaal Luchtvaartlaboratorium, Amsterdam. Report **F. 76**, i+27+27 pp. (1951).

The lifting surface method developed by the author in previous reports [see the preceding review] being too elaborate for practical use, he now presents a simpler method based on a generalization of Prandtl's lifting line theory for the straight wing under yaw, and for the symmetrical swept-back wing. After showing how Prandtl's equation for the wing under zero yaw may be derived from lifting surface theory by appropriate approximations, the author starts with a similar substitution for the distribution of the spanwise components of the vorticity over the lifting surface. The chordwise component then follows from Helmholtz' theorem. An oblique coordinate system is introduced. The resulting integro-differential equation for the wing under yaw is shown to reduce to Prandtl's equation for vanishing yaw. With the swept-back wing two cases have to be distinguished: pivotal point far from median section, and pivotal point on the median section. Continuity of the vorticity distribution along the root chord is ensured by special considerations. *H. Parkus* (Lansing, Mich.).

van Heemert, A. Application of the generalized Prandtl equation to an elliptic plane wing under yaw. Nationaal Luchtvaartlaboratorium, Amsterdam. Report **F. 77**, i+12+3 pp. (1951).

As a check on his generalization of Prandtl's lifting line equation [see the preceding review], the author applies his method to the case of an elliptic wing under yaw, where the exact solution has been given by Krienes. An axes-ratio of 5:1 and an angle of yaw of 30° are assumed. The difference between the exact values and the approximations is quite small for the derivative of the lift coefficient and the pitching moment coefficient, and is of the same order as the difference between the result of Prandtl's approximation and the exact value for the elliptic wing under zero yaw. No agreement at all, however, is found for the derivative of the rolling moment coefficient. This result becomes even worse if the contributions from the wing tips are taken into consideration with increased weight. *H. Parkus* (Lansing, Mich.).

Holme, O. A numerical solution of the problem of the lifting surface with continuous distribution of vorticity. Flygtekn. Försöksanstalt. Rep. no. **32**, 20 pp. (1950).

Lapin, E., Crookshanks, R., and Hunter, H. F. Downwash behind a two-dimensional wing oscillating in plunging motion. J. Aeronaut. Sci. **19**, 447–450, 458 (1952).

The downwash, in magnitude and phase, at various distances directly behind a two-dimensional thin airfoil oscillating in plunging motion is calculated on the basis of unsteady-airfoil theory. Results are obtained in closed form and are presented graphically. *W. R. Sears.*

van de Vooren, A. I. Generalization of the Theodorsen function to stable oscillations. J. Aeronaut. Sci. **19**, 209–211 (1952).

Laitone, E. V. Theodorsen's circulation function for generalized motion. J. Aeronaut. Sci. **19**, 211–213 (1952).

Jones, W. P. The generalized Theodorsen function. J. Aeronaut. Sci. **19**, 213 (1952).

These three papers are concerned with the difficulties that arise when Theodorsen's analysis [Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **496** (1935)] of monochromatic motion of an airfoil is applied to a damped oscillatory motion, as proposed by Luke and Dengler [same J. **18**, 478–483 (1951); these Rev. **13**, 77] in calculating the Theodorsen function $C(k)$ for $\mathrm{Im}\, k>0$. As pointed out by Jones, the latter problem cannot be treated in any proper sense without introducing initial conditions, the application of which to a truly monochromatic motion is impossible. Van de Vooren extends the analysis to a damped (either positively or negatively) oscillating motion, having a finite history. He also shows that a stability analysis requires a knowledge of $C(k)$ only in $\mathrm{Im}\, k<0$. Laitone, following Sears [J. Franklin Inst. **230**, 95–111 (1940); these Rev. **2**, 28], applies the Laplace transform to the problem and also points out an error in a related paper of Garrick's [Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **629** (1938)]. Dengler, Goland and Luke [J. Aeronaut. Sci. **19**, 213–214 (1952)] have offered the defense that their method is "adequate for engineering

purposes", a conclusion based on numerical comparison of stability and response studies, but they did not effectively answer any of the arguments advanced against their work, particularly the conclusion that $C(k)$ for Im $(k)>0$ is not required in practical studies. *J. W. Miles.*

Schmeidler, Werner. Zur Theorie des Schwingenfluges. II. S.-B. Berlin. Math. Ges. **40–41**, 1–12 (1942).

The lifting surface integral equation for an oscillating wing is established with the aid of vortex concepts. This result is then reduced to a lifting line integral equation on the assumption that the wing chord is small relative to its span. Finally, the latter equation is further simplified by assuming the prescribed motion of the wing to depend only on the spanwise variable. Possible methods of numerical solution are discussed. [For a résumé of more recent work, see Reissner, Bull. Amer. Math. Soc. **55**, 825–850 (1949); Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1953** (1949); these Rev. **11**, 274, 273.] *J. W. Miles.*

Radok, J. R. M. The theory of aerofoils in unsteady motion. Aeronaut. Quart. **3**, 297–320 (1952).

Expository paper on the subject of two-dimensional incompressible non-steady airfoil theory, presented in a form suitable for application in connection with the analysis of gust loads on aircraft. *E. Reissner.*

Couchet, G. Existence des mouvements plans non stationnaires d'un profil à pointe dans lesquels les forces aérodynamiques ne travaillent jamais. Recherche Aéronautique no. **28**, 13–15 (1952).

Formulas for forces and moment acting on a profile, derivable from a circle, moving non-uniformly but with constant circulation in plane, inviscid, incompressible flow are written down. The condition that the forces and moment do zero work, together with the condition for constant circulation according to the Kutta condition, leads to a specification for the motion. An example involving a flat-plate airfoil is worked out. *W. R. Sears* (Ithaca, N. Y.).

Kawaguti, Mitutosi. On the viscous shear flow around a circular cylinder. Rep. Inst. Sci. Tech. Univ. Tokyo **6**, 85–91 (1952). (Japanese. English summary)

Viscous shear flows around a circular cylinder are studied to the second Stokes' approximation. With Stokes' approximation, the boundary condition at infinity cannot be satisfied, so that in this paper the condition is replaced by the requirement that the viscous and perfect fluid solutions should be smoothly joined with each other on a circle $r=R$ (R being large but finite). *From the author's summary.*

Taylor, Geoffrey. The action of waving cylindrical tails in propelling microscopic organisms. Proc. Roy. Soc. London. Ser. A. **211**, 225–239 (1952).

In an earlier paper [same Proc. **209**, 447–461 (1951); these Rev. **13**, 596] the author presented a theory of the self-propulsion of microorganisms in which the tail of the organism was idealized as a thin waving sheet influenced only by viscous stresses in the surrounding medium. In order to render the theory quantitatively more realistic the author assumes in the present paper that the tail of the microorganism is a flexible cylinder bent in the form of transverse waves which propagate along its length. The mathematical account, in which again the inertial terms in the equations are ignored and various reasonable simplifications adopted, shows that waves propagated backwards along the tail result in a forward motion with velocity proportional to the square of the ratio of the amplitude of the waves to their length. Similar calculations for the case when the waves are propagated as spirals down the tail show that the body is propelled at twice the speed given it by transverse plane waves of the same amplitude and that in addition there is a resultant torque on the body which would tend to make it rotate. The author describes an experiment on a working model of a spermatozoon. Although the model swam, the quantitative results were inconclusive, presumably because of the crudeness of the model. *D. Gilbarg.*

Cooke, J. C. On Pohlhausen's method with application to a swirl problem of Taylor. J. Aeronaut. Sci. **19**, 486–490 (1952).

The first part of the paper is a critique of the method used by some authors in applying the Kármán-Pohlhausen approximation to three-dimensional cases. Specifically, they have assumed that the boundary layers of the different velocity components reach free-stream conditions at the same height above the surface. (This is a question peculiar to the Kármán-Pohlhausen treatment, which does not arise when the boundary-layer equations are solved exactly.) It is shown, on the basis of some exact solutions, that this assumption is unjustified.

The author then proceeds to test the importance of this assumption by means of examples. In the first example, viz., von Kármán's rotating disc [Z. Angew. Math. Mech. **1**, 233–252 (1921), pp. 244–247], the solution with unequal thicknesses is not quite as good as von Kármán's where equal thicknesses were assumed. The well-known swirl-atomizer problem of Taylor [Quart. J. Mech. Appl. Math. **3**, 129–139 (1950); these Rev. **11**, 697] is then solved anew, with unequal thicknesses. There are significant differences between these results and Taylor's. As the author states, it is not certain in this case which are better.

W. R. Sears (Ithaca, N. Y.).

Schlichting, H., und Truckenbrodt, E. Die Strömung an einer angeblasenen rotierenden Scheibe. Z. Angew. Math. Mech. **32**, 97–111 (1952). (German. English, French and Russian summaries)

This is the axisymmetric case of a rotating plane disc impinged upon by a viscous flow normal to it. In the limiting case of vanishing stream speed, the well-known problem studied by von Kármán and by Cochran [Proc. Cambridge Philos. Soc. **30**, 365–375 (1934)] is obtained. At the other extreme, when the disc does not rotate, the axisymmetric stagnation-point flow of Homann [same Z. **16**, 153–164 (1936); Forschung Gebiete Ingenieurwesens. Ausg. B. **7**, 1–10 (1936)] is realized. The same laminar boundary-layer approximations are used as in von Kármán's and Homann's investigations. Following von Kármán, the momentum-integral equations for radial and azimuthal directions are written and are attacked by the familiar Kármán-Pohlhausen procedure. As first approximation, von Kármán's choice of 4th- and 3rd-degree polynomials for the radial and azimuthal velocity profiles is adopted. As a second approximation, 5th- and 4th-degree polynomials are used. Results, including boundary-layer thickness, wall shearing stress, torque on the disc, and velocity profiles, are tabulated and plotted as functions of the significant parameter a/ω. Here ω is the angular velocity of the disc and ar is the radial velocity component of the potential stagnation-point flow (outside the boundary layer) at radius r. The numerical values depend greatly on this ratio and generally agree well

with Cochran's and Homann's results for $a=0$ and $\omega=0$, respectively. *W. R. Sears* (Ithaca, N. Y.).

Thwaites, B. On the momentum equation in laminar boundary-layer flow. A new method of uniparametric calculation. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2587** (11,155), 9 pp. (1952).

It is well known that Pohlhausen's method of solving laminar boundary layer flow with pressure gradient fails when the parameter, formed by the product of the square of the boundary layer thickness and the velocity gradient, exceeds the value 12. To remove this difficulty, the author's scheme is to take the normal coordinate y as dependent variable and the velocity component u as independent variable. For each family of velocity distribution, the functional relation between y and u must satisfy a normalization condition from the definition of the momentum thickness θ. Several forms of such functions have been proposed for both the solid wall and the porous wall. For the case of solid wall with pressure gradient, a convenient form is suggested as:

$$y/\theta=B(t)\left[1-\int_0^t(2t-1)G(t,\lambda)dt\right]+G(t,\lambda)$$

where $B(t)$ gives Blasius profile, λ is a parameter, $t=u/U$, U being the velocity on the edge of the boundary layer. The function $G(t,\lambda)$ is selected to give separation at $\lambda=1$, the shear is positive and ∂^2u/∂^2y on the wall can be both negative and positive. Such a $G(t,\lambda)$ with all these conditions satisfied is given explicitly. Once this function is chosen, the momentum integral equation can be easily integrated step-by-step. *Y. H. Kuo* (Ithaca, N. Y.).

Wuest, Walter. Grenzschichten an zylindrischen Körpern mit nichtstationärer Querbewegung. Z. Angew. Math. Mech. **32**, 172–178 (1952). (German. English, French and Russian summaries)

The principle of "independence" of the transverse and axial flow components, in the case of laminar boundary-layer flow about an infinite cylinder placed in a uniform stream oblique to its axis, holds as well in steady and unsteady flow. The author considers certain problems for which the transverse flow pattern is steady while the axial component is variable with time, because of transient axial motion of the cylinder, for example. The cases treated are principally the well-known Falkner-Skan-Hartree ones, for which the transverse potential-flow component is proportional to a power of the distance along the surface. The axial potential velocity is taken to be exponential or sinusoidal with time. For two simple cases numerical results are given. The case of flow around a circular cylinder is discussed briefly. *W. R. Sears* (Ithaca, N. Y.).

Young, George B. W., and Janssen, Earl. The compressible boundary layer. J. Aeronaut. Sci. **19**, 229–236, 288 (1952).

The authors solve the compressible boundary layer equations for the flat plate on the mechanical differential analyzer by utilizing the best known variation of the air properties with temperature for both insulated-plate and heat-transfer cases. The velocity and temperature distributions over a range of Mach numbers up to 15.95 are presented. For the case of insulated plate, the skin friction coefficient is compared with the results of von Kármán and Tsien [same J. **5**, 227–232 (1938)]. The deviation begins to show at Mach number about 6, but at Mach number 10 the difference amounts only to about 10 per cent. This seems to indicate that outside the hypersonic range a refinement as herein proposed is indeed insignificant. *Y. H. Kuo.*

Bass, Jean. Les équations générales des corrélations spatio-temporelles dans un fluide turbulent. C. R. Acad. Sci. Paris **234**, 806–808 (1952).

In the current statistical theories of turbulence one defines correlations such as $\overline{u_i(\mathbf{r}_1,t)u_j(\mathbf{r}_2,t)}$ of the instantaneous components of the velocity at two different points in the fluid. In this paper the author extends these concepts to include correlations of velocity components at different instants of time, e.g., $\overline{u_i(\mathbf{r}_1,t_1)u_j(\mathbf{r}_2,t_2)}$. The forms which these correlation tensors should have for an incompressible fluid under conditions of isotropy can be readily written down from the theory of solenoidal isotropic tensors. The author next defines turbulence as stationary if the correlations considered depend, in addition to $\xi=\mathbf{r}_1-\mathbf{r}_2$, only on t_2-t_1. The form which the Stokes-Navier equation takes in this case is also written down. *S. Chandrasekhar.*

Bass, Jean. La structure locale des corrélations spatio-temporelles dans un fluide turbulent. C. R. Acad. Sci. Paris **234**, 1033–1034 (1952).

By expanding the scalars defining the tensors such as $R_{ij}(\xi,\tau)=\overline{u_i(\mathbf{r}_1,t)u_j(\mathbf{r}_2,t+\tau)}$ introduced in the paper reviewed above, the author shows that the concept of stationary turbulence leads to physically inconsistent results except when the kinematic viscosity is zero.

S. Chandrasekhar (Williams Bay, Wis.).

Bass, Jean. Sur un type d'écoulement turbulent non homogène. C. R. Acad. Sci. Paris **234**, 2256–2257 (1952).

In this paper the author considers the form of the correlation tensor $\overline{u_i(\mathbf{r}_1,t)u_j(\mathbf{r}_1+\xi,t+\tau)}$ in case the turbulence is axisymmetric [see the two preceding reviews; also Batchelor, Proc. Roy. Soc. London. Ser. A. **186**, 480–502 (1946); and Chandrasekhar, Philos. Trans. Roy. Soc. London. Ser. A. **242**, 557–577 (1950); these Rev. **8**, 238; **12**, 368].

S. Chandrasekhar (Williams Bay, Wis.).

***Coburn, N. A method for constructing correlation tensors in homogeneous turbulence.** Proceedings of the Midwestern Conference on Fluid Dynamics, 1950, pp. 129–141. J. W. Edwards, Ann Arbor, Michigan, 1951.

In the statistical theory of turbulence one needs to construct tensors such as $\overline{u_iu_j'}$ representing the correlation of the velocity components u_i and u_j' at two points P and P' separated by a distance vector ξ_i. When one says that this tensor is isotropic, one means that $a_ib_j\overline{u_iu_j'}$, where a_i and b_i are arbitrary unit vectors at P and P' respectively, is a scalar invariant for all rotations (proper and improper) of the vector configuration ξ, $\mathbf{a}$, $\mathbf{b}$ as a rigid body; and it follows from a theorem on scalar invariants that $a_ib_j\overline{u_iu_j'}$ must be a function only of the fundamental invariants ξ_ia_i, a_ib_j and $\epsilon_{ijk}a_ib_j\xi_k$. From this, the most general form of $\overline{u_iu_j'}$ can be readily deduced. This is the method of Robertson [Proc. Cambridge Philos. Soc. **36**, 209–223 (1940); these Rev. **1**, 286]. In this paper the author proceeds in a somewhat different way. Let λ and μ be two unit vectors which together with ξ form an orthogonal set of vectors. Resolve u_i and u_j' along these directions and form the tensor $\overline{u_iu_j'}$. Now impose the condition that $\overline{u_iu_j'}$, when expressed in

terms of ξ/r, λ, and μ, is invariant for rotations and reflections of λ and μ in planes perpendicular to ξ and, in particular, for the transformation $\lambda_i \to -\lambda_i$ and $\mu_i \to -\mu_i$. We then recover the result obtained by the method of invariants. The method is capable of generalization for obtaining tensors having other invariant properties. Thus if n is some fixed direction and λ is a direction orthogonal to ξ and n, we can require that $\overline{u_i u_j'}$ is invariant for the transformation $\eta_i \to -\eta_i$. The forms of tensors with other similar invariant properties are also derived. *S. Chandrasekhar.*

Frenkiel, François N. On the statistical theory of turbulent diffusion. Proc. Nat. Acad. Sci. U. S. A. **38**, 509–515 (1952).

It is well known that if a particle moves in such a way that its velocity at two instants t_1 and t_2 are correlated in the manner $\overline{u_i(t)u_j(t+\tau)} = \overline{u^2}R(\tau)$, the mean dispersion in the position of the particle at time t, if it was known to have been at the origin at time $t=0$, is given by

$$(1) \qquad \overline{x_i^2} = 2\overline{u^2}\int_0^t (t-\tau)R(\tau)\,d\tau = \Delta(t) \quad \text{(say)}.$$

On the other hand, if a large number of particles, Q, is "realized" at the origin at time $t=0$, then the distribution of the particles at a later time may be expected to be given by

$$(2) \qquad S(x, y, z, t) = \frac{Q}{(2\pi\Delta)^{3/2}} e^{-(x^2+y^2+z^2)/2\Delta}$$

where Δ is given by (2). The author considers various special consequences of combining (1) and (2).

S. Chandrasekhar (Williams Bay, Wis.).

***v. Krzywoblocki, M. Z. On the fundamentals of kinematics of statistical theories of turbulence in compressible fluids.** Proceedings of the Midwestern Conference on Fluid Dynamics, 1950, pp. 66–80. J. W. Edwards, Ann Arbor, Michigan, 1951.

This paper attempts to formulate in a "fundamental way" the concepts of spatial and temporal averages which underlie the current statistical theories of turbulence.

S. Chandrasekhar (Williams Bay, Wis.).

Corrsin, Stanley. Heat transfer in isotropic turbulence. J. Appl. Phys. **23**, 113–118 (1952).

If the temperature (T) and the velocity (U_k) in an incompressible fluid are subject to fluctuations, then writing $T = \bar{T} + \vartheta$ and $U_k = \bar{U}_k + u_k$ where $\bar{T}$ and $\bar{U}_k$ are the mean values and ϑ and u_k are the fluctuations, we obtain on averaging the equation of heat conduction,

$$\frac{\partial}{\partial t}(\bar{T}+\vartheta) + \frac{\partial}{\partial x_k}(\bar{T}+\vartheta)(\bar{U}_k+u_k) = \kappa\nabla^2(\bar{T}+\vartheta),$$

where κ is the coefficient of thermometric conductivity, and

$$\frac{\partial \bar{T}}{\partial t} + \frac{\partial}{\partial x_k}\bar{T}\bar{U}_k + \frac{\partial}{\partial x_k}(\overline{\vartheta u_k}) = \kappa\nabla^2\bar{T}.$$

The fluctuations in temperature are therefore governed by

$$\frac{\partial \vartheta}{\partial t} + \bar{U}_k\frac{\partial \vartheta}{\partial x_k} + u_k\frac{\partial \bar{T}}{\partial x_k} + \frac{\partial}{\partial x_k}(\vartheta u_k) - \frac{\partial}{\partial x_k}(\overline{\vartheta u_k}) = \kappa\nabla^2\vartheta.$$

Multiplying this equation by ϑ and averaging we have

$$\left(\frac{\partial}{\partial t} + \bar{U}_k\frac{\partial}{\partial x_k}\right)\overline{\vartheta^2} = -2\overline{\vartheta u_k}\frac{\partial \bar{T}}{\partial x_k} - \frac{\partial}{\partial x_k}(\overline{u_k\vartheta^2}) + \kappa\nabla^2\overline{\vartheta^2} - 2\kappa\overline{\left(\frac{\partial \vartheta}{\partial x_i}\frac{\partial \vartheta}{\partial x_i}\right)}.$$

This is one of the basic equations derived by the author. A corresponding Lagrangian analysis based on Taylor's theory of diffusion by continuous movements [Proc. London Math. Soc. **20**, 196–211 (1921)] is also given. Based on these equations, the author draws a number of conclusions of a dimensional character. *S. Chandrasekhar.*

Moyal, J. E. The spectra of turbulence in a compressible fluid; eddy turbulence and random noise. Proc. Cambridge Philos. Soc. **48**, 329–344 (1952).

Expressing the instantaneous velocity $u_i(\mathbf{r}, t)$ in a homogeneous turbulent medium as a Fourier integral in the form

$$u_i(\mathbf{r}, t) = \int e^{i\mathbf{k}\cdot\mathbf{r}}\,dZ_i(\mathbf{k}, t),$$

the author first shows that the spectral function $dZ_i(\mathbf{k}, t)$ can be resolved into a transverse ($dZ_i^{(t)}(\mathbf{k})$) and a longitudinal ($dZ_i^{(l)}(\mathbf{k})$) component in directions perpendicular and parallel (respectively) to the wave vector $\mathbf{k}$; further, he shows that in isotropic turbulence the two components are uncorrelated. Resolving $d\mathbf{Z}^{(t)}(\mathbf{k})$ along two directions specified by the unit vectors $\alpha(\mathbf{k})$ and $\beta(\mathbf{k})$ such that its components $dZ_\alpha(\mathbf{k})$ and $dZ_\beta(\mathbf{k})$ along these directions are uncorrelated, we can write

$$d\mathbf{Z}(\mathbf{k}) = \alpha(\mathbf{k})dZ_\alpha(\mathbf{k}) + \beta(\mathbf{k})dZ_\beta(\mathbf{k}) + (\mathbf{k}/k)dZ_k(\mathbf{k}).$$

If $\phi_{ij}(\mathbf{k}, t)$ is the Fourier transform of the fundamental correlation tensor $T_{ij}(\mathbf{r}, \mathbf{r}', t) = \overline{u_i(\mathbf{r}, t)u_j(\mathbf{r}', t)}$, then the corresponding resolution of ϕ_{ij} is

$$\phi_{ij} = \psi_{ij} + \eta_{ij} + \rho_{ij}$$

where

$$\psi_{ij}(\mathbf{k}) = \overline{dZ_i^{(t)*}dZ_j^{(t)}}/d\mathbf{k} = \psi(\mathbf{k})\{\delta_{ij} - k_ik_j/k^2\} + \theta_{ij}(\mathbf{k}),$$

$$\eta_{ij}(\mathbf{k}) = \overline{dZ_i^{(k)*}dZ_j^{(k)}}/d\mathbf{k} = 2\eta(\mathbf{k})k_ik_j/k^2,$$

$$\rho_{ij}(\mathbf{k})d\mathbf{k} = \overline{dZ_i^{(t)*}dZ_j^{(k)}} + \overline{dZ_j^{(t)*}dZ_i^{(k)}}$$

$$\psi(\mathbf{k})d\mathbf{k} = \tfrac{1}{2}\{\overline{|dZ_\alpha|^2} + \overline{|dZ_\beta|^2}\}, \quad \eta(\mathbf{k})d\mathbf{k} = \tfrac{1}{2}\overline{|dZ_k|^2}$$

and

$$\theta_{ij}(\mathbf{k})d\mathbf{k} = \tfrac{1}{2}\{\overline{|dZ_\alpha|^2} - \overline{|dZ_\beta|^2}\}\{\alpha_i^*\alpha_j - \beta_i^*\beta_j\}.$$

In isotropic turbulence $\theta_{ij}(\mathbf{k})$ and $\rho_{ij}(\mathbf{k})$ are identically zero and the spectral tensor becomes simply the sum of the transverse and the longitudinal components ψ_{ij} and η_{ij}. Corresponding to this resolution of ϕ_{ij} we can resolve T_{ij} into a transverse and a longitudinal component. This unique resolution of the various correlation and spectral tensors corresponds to the distinct physical phenomena associated with eddy turbulence and disordered acoustical waves or random noise, respectively. Various formal relations between the scalars defining the correlation tensor T_{ij} and the spectral tensor ϕ_{ij} resulting from this resolution are explicitly worked out. Considering the Fourier transform of the equations of continuity, motion and heat transfer, the author shows in particular that the equation governing the transverse spectral tensor $\psi_{ij}(\mathbf{k})$ is

$$\frac{\partial \psi_{ij}(\mathbf{k})}{\partial t} + 2\nu k^2\psi_{ij}(\mathbf{k}) = \alpha_{ij}(\mathbf{k}) + \beta_{ij}(\mathbf{k})$$

where

$$\alpha_{ij}(\mathbf{k})d\mathbf{k}=\overline{dZ_i^{(0)*}dC_j(\mathbf{Z})}+\overline{dZ_j^{(0)}dC_i^*(\mathbf{Z})}$$

$$\beta_{ij}(\mathbf{k})=-(k_n/k^2)\{k_i\alpha_{jn}+k_j\alpha_{ni}\}$$

and

$$dC_i(\mathbf{Z})=\sqrt{(-1)}\int k_j'dZ_j^*(\mathbf{k}'-\mathbf{k})dZ_i(\mathbf{k}').$$

(In the last equation the integration is over $\mathbf{k}'$.) The author compares these equations with the corresponding ones valid for an incompressible fluid. *S. Chandrasekhar.*

Csada, I. K. On the magnetic effects of turbulence in ionized gases. Acta Phys. Acad. Sci. Hungar. 1, 235–246 (1952). (Russian summary)

The equations of hydromagnetics for an incompressible fluid are

$$(1)\quad \frac{\partial u_i}{\partial t}+\frac{\partial}{\partial x_j}u_iu_j-\frac{\mu}{4\pi\rho}\frac{\partial}{\partial x_j}H_iH_j=-\frac{1}{\rho}\frac{\partial}{\partial x_i}\left(p+\frac{\mu|\mathbf{H}|^2}{8\pi}\right)+\nu\nabla^2u_i,$$

and

$$(2)\quad \frac{\partial H_i}{\partial t}+\frac{\partial}{\partial x_j}(H_iu_j-H_ju_i)=\eta\nabla^2H_i\quad(\eta=1/4\pi\sigma),$$

where μ and σ are the coefficients of magnetic permeability and electrical conductivity respectively and the rest of the symbols have their usual meanings. The author treats these equations in the manner in which the Stokes-Navier equation (i.e., eq. (1) with $H_i=0$) has been treated in the past for defining the Reynolds stresses. Thus, writing $H_i=\bar{H}_i+\delta H_i$ and $u_i=\bar{u}_i+\delta u_i$ where $\bar{H}_i$ and $\bar{u}_i$ are the mean values and δH_i and δu_i are the instantaneous fluctuations, we find that the averaged form of equation (2) is

$$(3)\quad \frac{\partial \bar{H}_i}{\partial t}+\frac{\partial}{\partial x_j}(\bar{H}_i\bar{u}_j-\bar{H}_j\bar{u}_i)=\eta\nabla^2\bar{H}_i-\frac{\partial}{\partial x_j}(\overline{\delta H_i\delta u_j}-\overline{\delta H_j\delta u_i}).$$

With the further assumptions

$$(4)\quad \delta H_i=-l_k\frac{\partial \bar{H}_i}{\partial x_k}\quad\text{and}\quad\overline{l_j\delta u_i}=-\Lambda\delta_{ij}$$

where l_k is a vector defining the process associated with the concept of the "mixing length" and $\Lambda(=\overline{|l|\cdot|u|})$ is a constant, equation (3) becomes

$$(5)\quad \frac{\partial \bar{H}_i}{\partial t}+\frac{\partial}{\partial x_j}(\bar{H}_i\bar{u}_j-\bar{H}_j\bar{u}_i)=(\eta+\Lambda)\nabla^2\bar{H}_i.$$

The mean magnetic intensity therefore satisfies an equation of the same form as (2) but with a different η; in other words we may define an "eddy-electrical conductivity" just as an eddy-viscosity has been defined in the "heuristic" theories of turbulence. *S. Chandrasekhar* (Williams Bay, Wis.).

Kapica, P. L. Heat conduction and diffusion in a fluid medium with a periodic flow. I. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. 21, 964–978 (1951). (Russian)

Consideration is given to the increase in effective thermal conductivity in the mean heat flow in an oscillating fluid medium. The increment of relative increase, termed the coefficient of wave transfer, is investigated for flow in tubes, slits, and channels. For the case of laminar flow in a circular tube with assumed periodic parabolic distribution, the wave transfer coefficient is determined in terms of series involving Bessel functions and zeros of Bessel functions. For the other configurations familiar solutions are obtained. The analysis does not appear to be significantly fundamental, but does present several ingenious devices to handle the problem. *N. A. Hall* (Minneapolis, Minn.).

***Sauer, Robert. Einführung in die theoretische Gasdynamik.** 2d ed. Springer-Verlag, Berlin, Göttingen, Heidelberg, 1951. viii+174 pp.

This volume remains very similar to its first edition [these Rev. 7, 92; 10, 855] but contains improvements in the exposition, some additional material, and several new references to recent literature. The most important additions are a chapter on spatial non-axially symmetric supersonic flow (including finite thin wing theory and linearized conically symmetric flow) and a broadened section on the method of characteristics to take account of rotational flow. There is also a passing glance in a half page at Bergman's operator method. The presentation of the linearized theory (chapter 2) is improved and corrects some errors. The reviewer notes that the treatment of the relation between entropy and vorticity (p. 144) actually presupposes isoenergetic flow but gives the erroneous impression (not present in the first edition) that the conclusions are more general. *D. Gilbarg* (Bloomington, Ind.).

Ericksen, J. L. On the uniqueness of gas flows. J. Math. Physics 31, 63–68 (1952).

This paper shows that if two steady plane irrotational flows of a polytropic gas have the same streamline pattern, then either the two velocity fields are strictly proportional and the two gases have the same adiabatic exponent, or the two flows are of the special source-sink type (streamlines radial lines) or of the vortex type (streamlines concentric circles or parallel lines). Hence the plane irrotational gas flows are (essentially) uniquely determined by their geometry alone. This extends to the compressible case analogous results of the reviewer and Prim on incompressible fluids. *D. Gilbarg* (Bloomington, Ind.).

Prim, R. C., 3rd. Steady rotational flow of ideal gases. J. Rational Mech. Anal. 1, 425–497 (1952).

This paper concerns the steady rotational flow of nonviscous and thermally nonconducting gases with attention for the most part restricted to perfect gases and gases with a product equation of state $p=P(\rho)S(s)$. It is intended as a comprehensive treatment of rotational flow theory (up to 1949) from the point of view of (A) the establishment of general dynamical properties of such flows and (B) the discovery of families of exact solutions to the equations governing them. Related to (B) is (C) the study of the existence of flows with given streamline patterns. Although many of the results have already been published piecemeal, part of the work appears for the first time in accessible form. The author's procedure and motivation depend on the Munk-Prim substitution principle and upon the so-called canonical equations of motion. The following topics are considered.

(A) It is known (Munk-Prim principle) that there exists a multiplicity of flows with a given streamline pattern. The velocity vector $\mathbf{W}$ of that one of these flows with unit ultimate speed on each streamline is called the reduced velocity vector, the canonical equation being the equations satisfied by $\mathbf{W}$. A solution of the canonical equation generates a set of flows having the same streamline pattern. Crocco's pressure theorem is proved for the reduced velocity

field, and Crocco's equation for the streamfunction is derived without the usual restrictions on the flow. The concept of Beltrami flow fields ($\mathbf{V}\times\operatorname{curl}\mathbf{V}=0$) is generalized to $\mathbf{W}\times\operatorname{curl}\mathbf{W}=0$, and the original work of Beltrami [Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (2) **22**, 121–130 (1889)] is thereby extended. Poritsky's superposition principle is treated from the point of view of the reduced velocity field, leading to a much larger variety of spatial flows obtainable from a given plane flow.

(B) The author determines all plane flows for which (1) $\mathbf{W}=\mathbf{W}(\theta)$ (generalized Prandtl-Meyer flows) and (2) $\mathbf{W}=iu(x)+jv_1(x)v_2(y)$, and all axi-symmetric flows for which $\mathbf{W}=\mathbf{W}(r)$. It is determined that the only plane flows having constant speed on streamlines are vortex flows, that the only axi-symmetric flows having this property are uniform, and that the only doubly laminar flows having this property are helicoidal, the latter generalizing a result of Hamel. The reviewer notes that the proofs have been improved upon since 1949, and that the assumption of a product equation of state can be avoided.

(C) Plane flows whose streamlines are straight are shown to be simple source flows. The corresponding problem for spatial flows was considered by Merlin [Ann. Sci. École Norm. Sup. (3) **55**, 223–255 (1938)]. Plane flows whose streamlines and orthogonal trajectories form an isometric net are shown to be vortex or simple source flows.

J. B. Serrin (Cambridge, Mass.).

Cabannes, Henri. Etude de l'onde de choc attachée dans les écoulements de révolution. II. Cas d'un obstacle terminé par un cône. Recherche Aéronautique no. **27**, 7–16 (1952).

(Two brief reports were published earlier [C. R. Acad. Sci. Paris **231**, 325–326 (1950); **232**, 481–483 (1951); these Rev. **12**, 299, 767].) The conical nose of a body of revolution is supposed to be joined to a non-conical afterbody. The whole is at zero incidence in a stream of sufficiently great Mach number M to produce an attached shock. Two cases are considered; viz., the flow behind the attached shock is (a) subsonic or (b) supersonic.

In case (a) the effect of the joining to the afterbody influences the entire flow field behind the shock. The shock has infinite curvature at the nose. As an example, one special case (20° cone, $M=1.2175$) is worked out and the shape of the shock is plotted. The singularity is sufficiently weak so that the calculated shock gives the appearance of tangency to the conical shock that would occur for an infinite cone. The exponent of the singularity at the nose is given for 20° cones at various Mach numbers. These results help to explain a certain paradox encountered in Part I [same journal, no. **24**, 17–23 (1951); these Rev. **13**, 597; see also the second reference mentioned above].

In case (b) there is a conical flow field behind the shock, terminated by a characteristic surface emanating from the point where the afterbody begins. The shock wave is conical back to its intersection with this characteristic surface. Here the geometry of the shock at the intersection is determined, and again a numerical example is provided. The limiting case is discussed in which the length of the conical nose is very small compared to the radius of curvature of the afterbody behind the cone.

A bibliography of papers concerning attached shocks in plane and axisymmetric flow is appended. No comparison with experiment is made here. The observations of Maccoll on bullets with conical heads [Proc. Roy. Soc. London. Ser. A. **159**, 459–472 (1937)] should be useful for such a comparison. *W. R. Sears* (Ithaca, N. Y.).

Manwell, A. R. A note on the hodograph transformation. Quart. Appl. Math. **10**, 177–184 (1952).

Solutions of the hodograph equations have, of course, only limited value when the hodograph plane cannot be mapped regularly into the physical plane. The author establishes as a test for this state of affairs the result that if the boundary streamline can be so mapped, then so can the whole field. Arguments indicating that this was so were given previously [Craggs, Proc. Cambridge Philos. Soc. **44**, 360–379 (1948); Friedrichs, Communications on Appl. Math. **1**, 287–301 (1948); these Rev. **10**, 640, 638] but the author claims that his result is more general. There is some physical discussion (not quite clear to the reviewer) of its implications.

M. J. Lighthill (Manchester).

Ting, Lu. The shock strength in a two-dimensional nonsteady flow. J. Aeronaut. Sci. **19**, 351–352 (1952).

The reviewer's technique for improving approximate solutions near singularities [Philos. Mag. (7) **40**, 1179–1201 (1949); these Rev. **11**, 518] is applied to obtain the distribution of shock strength around the Mach cone in a general two-dimensional unsteady disturbance which starts at the origin at time $t=0$. The author discusses in particular the strength of the reflected shock in Mach reflection and obtains results in agreement with those of Tan [J. Aeronaut. Sci. **18**, 768–770 (1951); these Rev. **13**, 701].

M. J. Lighthill (Manchester).

Chu, Boa-Teh. On weak interaction of strong shock and Mach waves generated downstream of the shock. J. Aeronaut. Sci. **19**, 433–446 (1952).

The flow with a uniform attached shock wave produced by a wedge in a steady uniform supersonic stream is perturbed by supposing that the surface of the wedge departs slightly from the straight. Then the pressure and flow direction satisfies the ordinary linearised equations of motion relative to the flow behind the shock (even though, due to rotationality, the longitudinal velocity component does not), and also a certain boundary condition at the shock. This imposes a certain reflection coefficient at the shock, and the solution consists of pressure waves emitted from the surface according to its shape and reflected from the shock with this reflection coefficient. This coefficient is small except for shocks near the maximum strength compatible with attachment. A protuberance on the wedge surface is shown to produce a similar protuberance on the shock, only stretched in certain ratios both longitudinally and laterally.

All the above results were given previously by the reviewer [Philos. Mag. (7) **40**, 214–220 (1949); these Rev. **10**, 641] in a paper not cited by the author. The author goes on to discuss the multiple reflection of waves between wedge and shock (with attenuation on each reflection from the shock), and also the influence on the pressure distribution on the lower surface of an airfoil at high angle of attack of the pressure wave reflected from the shock.

M. J. Lighthill (Manchester).

Ting, L., and Ludloff, H. F. Aerodynamics of blasts. J. Aeronaut. Sci. **19**, 317–328 (1952).

The authors consider an initially uniform plane shock, of arbitrary strength, as it moves over surfaces all of which are inclined at a small angle to its initial direction of motion, and are such that the motion is two-dimensional. Then, in

a frame of reference in which the fluid behind the shock is at rest in the undisturbed state, the pressure satisfies the equation of sound (although, owing to the rotationality of the flow, the velocity components do not). In addition to the ordinary boundary condition at solid surfaces (which, in terms of the pressure, amounts to prescribing its normal derivative), there is a rather complicated condition holding along the straight line where the shock would be if it had not been disturbed (in fact, a certain second-order differential operator applied to the pressure vanishes there). Owing to the differential character of this condition (which implies that certain necessary constants have been eliminated) it is necessary also to impose continuity of the lateral pressure derivative at the point of intersection of the shock at the wall. When this is done, there is a unique solution under given initial conditions and conditions at infinity.

Since the shock boundary condition is one holding on a moving surface, the authors use a Lorentz transformation to reduce this surface to rest while retaining the wave equation for the pressure: this considerably simplifies the shock condition while complicating the others only slightly. Now the wall boundary conditions can be satisfied by a distribution of sources along the wall behind the shock. It is found possible in addition to satisfy the shock boundary condition if suitable sources are placed along the wall in front of the shock. Their strength is a linear combination of the source strengths at three different points behind the shock. In terms of the complete source distribution the pressure field can be calculated, and the density field can then be deduced from the condition that each particle of fluid retains the entropy which it achieved when it passed through the shock.

The density field resulting from diffraction at a single corner is calculated (the reviewer [Proc. Roy. Soc. London. Ser. A. **198**, 454–470 (1949); these Rev. **11**, 478] had previously obtained the shock shape and wall pressure distribution, by a method based on conical field theory, but the present method gives a relatively simple expression for the whole pressure field) and this agrees well with experiments made at the Palmer Physical Laboratory, Princeton University. *M. J. Lighthill* (Manchester).

Taunt, D. R., and Ward, G. N. Wings of finite aspect ratio at supersonic velocities. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2421** (**9401**), 12 pp. (1952).

This report deals with work completed in 1946. The error in Schlichting's analysis [Luftfahrtforschung **13**, 320–335 (1936)] of the supersonic rectangular wing is pointed out and the correct results derived using Lighthill's method [same Rep. and Memoranda no. **2001** (**8079**), (1944); these Rev. **8**, 108]. *J. W. Miles* (Los Angeles, Calif.).

Squire, H. B. An example in wing theory at supersonic speed. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2549** (**10,624**, **12,517**) 16 pp. (1951).

The examples treated are a flat elliptic cone and a flat "elliptic hypercone," both at zero incidence and lying completely inside the Mach cone from the apex. These can be combined to form a wing-like triangular surface with sharp trailing edge. The treatment follows Robinson [same Rep. and Memoranda no. **2548** (1946)], where a special system of coordinates was introduced and the potential found in terms of Lamé functions. The cone and hyper-cone yield two simple cases of this solution. The pressure over the surface of the cone is uniform; that over the surface of the hyper-cone is linear chordwise. The pressure distribution over the combined wing, which is also linear chordwise, is shown. The drag of the wing is then calculated, allowing for leading-edge singularities by the method of R. T. Jones [J. Aeronaut. Sci. **17**, 307–310 (1950); these Rev. **12**, 62]. *W. R. Sears* (Ithaca, N. Y.).

Miles, John W. A note on supersonic wing integral equations in unsteady flow. Aeronaut. Quart. **3**, 294–296 (1952).

"The integral equations for the pressure distribution on an oscillating thin wing having a prescribed velocity distribution in supersonic flow are developed. It was hoped to extend the methods available for the steady flow problem. All attempts to generalize these methods for wings of arbitrary plan form having proved nugatory it is feltth at the integral equations are worth recording." (From the author's summary.) *E. Reissner* (Cambridge, Mass.).

Muggia, Aldo. Sulla interferenza ala-fusoliera alle velocità iposoniche. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **85**, 27–40 (1951).

For incompressible flow Multhopp has devised methods for determining the spanwise lift distribution of a wing alone [Luftfahrtforschung **15**, 153–169 (1938)] and in the presence of a fuselage [ibid. **18**, 52–66 (1941); Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1036** (1942)] and also the downwash behind the wing [ibid. **15**, 463–467 (1938)]. The author has corrected these results for subsonic compressible flow by appropriate insertions of the Prandtl-Glauert factor $(1-M^2)^{1/2}$. He presents numerical results for a wing of aspect ratio 4.83 alone and in combination with two fuselages of circular cross-section at $M=0.3$, 0.6, and 0.8. *J. H. Giese* (Havre de Grace, Md.).

Young, A. D., and Kirkby, S. Application of the linear perturbation theory to compressible flow about bodies of revolution. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2624** (**11,033**), 8 pp. (1952).

Drougge, Georg. The flow around conical tips in the upper transsonic range. Flygtekn. Försöksanstalt. Rep. no. **25**, 16 pp. (8 plates) (1948).

J. W. Maccoll [Ministry of Supply, Armament Res. Dept., Theor. Res. Rep. No. **7/46**; presented to Sixth Int. Congress for Applied Mech., Paris, 1946] has described a relaxation method for computing steady non-viscous plane or axisymmetric flows about simple polygonal profiles with detached shocks, behind which entropy gradients are assumed negligible and the flow irrotational. He has also stated a similarity principle that on the subsonic parts of the profile $p/p_0=f(x)$, where p (p_0) is the local static (stagnation) pressure at the point x, and $f(x)$ depends on the profile but not on the Mach number of the undisturbed flow. Starting from shock shapes determined from schlieren photographs, the author has made such calculations of flows about a 45° half-angle cone-cylinder at Mach numbers 1.83 and 2.08. His wind tunnel data for this body agree well with the calculated pressure distributions and also confirm the similarity principle fairly well. *J. H. Giese.*

Behrbohm, Hermann. Näherungstheorie des unsymmetrischen Schalldurchgangs in einer Lavaldüse. Z. Angew. Math. Mech. **30**, 101–112 (1950). (German. English, French and Russian summaries)

In discussing the flow in the neighborhood of the throat of a two-dimensional subsonic-supersonic Laval nozzle, the

author linearizes the problem in a manner differing from that usually employed. Let $\varphi(x, y)$ and $\psi(x, y)$ denote the potential and stream functions of the isentropic flow through the nozzle such that $\varphi_x = u/c^*$; $\varphi_y = v/c^*$ and $\psi_x = -\rho v/\rho^* c^*$; $\psi_y = \rho u/\rho^* c^*$ where u and v are the x and y velocity components, ρ the density, c the local velocity of sound, and the star designates critical values. Then the hodograph transformation yields the well-known results

$$\frac{\partial \Theta}{\partial \varphi} = \frac{\partial W}{\partial \psi}, \quad \frac{\partial \Theta}{\partial \psi} = \frac{M^2 - 1}{(\rho/\rho^*)^2} \frac{\partial W}{\partial \varphi}$$

where $\tan \Theta = v/u$, $M = (u^2+v^2)^{1/2}/c = w/c$, and

$$W = \int_1^{w/c^*} \frac{\rho}{\rho^*} \frac{d(w/c^*)}{w/c^*}.$$

Since the Mach number and the density depend only upon w/c^*, it is clear that the quantity $M^2 - 1/(\rho/\rho^*)^2$ may be written $K(W)$. Near $w/c^* = 1$, $K(W)$ may be expressed $K(W) = (\kappa+1)W + \cdots$ and, defining a new potential ω: $W = \omega_\varphi$, $\Theta = \omega_\psi$, the hodograph equations become

$$\omega_{\psi\psi} - (\kappa+1)\omega_\varphi \omega_{\varphi\varphi} = 0,$$

the well-known quasi-linear partial differential equation which changes type with a change in sign of ω_φ.

Now the author effects an approximate linearization in the following manner: If the flow is nearly one-dimensional, then W is nearly independent of ψ; call the approximation $\bar{W}(\varphi)$. Then

$$\omega_{\varphi\varphi} \approx \frac{d\bar{W}}{d\varphi} \equiv G(\varphi) = \frac{\Theta_1(\varphi)}{(\kappa+1)(\psi_1 - \psi_0)} \frac{1}{\bar{W}(\varphi)}$$

where $\Theta_1(\varphi)$ is the angle of the nozzle wall, ψ_1 and ψ_0 are the values of the stream function on the wall and center line respectively. Changing variables to

$$\Phi \equiv \frac{1}{\kappa+1} \int_0^\varphi \frac{dt}{G(t)}; \quad \Psi \equiv \psi; \quad \Omega(\Phi, \Psi) \equiv \omega(\varphi, \psi)$$

the mathematical problem becomes, since $\Theta(\Phi, \Psi) = \Omega_\Psi(\Phi, \Psi)$,

$$\Theta_{\Psi\Psi} - \Theta_\Phi = 0,$$

a canonical parabolic partial differential equation, with the boundary and initial conditions

$$\begin{aligned} &\Theta(\Phi, 0) = 0 && \text{for } \Phi \geqq 0 \\ &\Theta(0, \Psi) = E(\Psi) && \text{for } 0 \leqq \Psi \leqq \Psi_1; \quad E(0) = 0 \\ &\Theta(\Phi, \Psi_1) = R(\Phi) && \text{for } \Phi \geqq 0; \quad R(0) = E(\Psi_1). \end{aligned}$$

The function $E(\Psi)$ corresponds to the prescription of flow angle at some point ahead of the nozzle throat. The author then discusses a particular solution of the equation and compares the results favorably with those of Oswatitsch and Rothstein [Jahrbuch der Deutschen Luftfahrtforschung 1942, pp. I91–I102; these Rev. 9, 391]. *F. E. Marble.*

Eliassen, Arnt. Slow thermally or frictionally controlled meridional circulation in a circular vortex. Astrophys. Norvegica 5, 19–60 (1952).

The atmosphere is considered to be a vortex with axis identical with the Earth's polar axis. Motion in a meridian plane is discussed under steady-state, or approximately steady-state, conditions. This motion is due to sources of heat or of angular momentum. The meridional motion is controlled by a generalized Poisson equation which is solved by means of a Green's function and leads to the conclusion that near a point-source of angular momentum, the streamlines are elliptical and the field of motion resembles that of a distorted dipole. Near a point-source of heat, a similar result holds good. Energy considerations are also dealt with. When applied to the atmosphere the theory leads to the conclusion that there are two main circulation cells in a meridional plane, the "trade wind cell" that extends from the earth's surface in low latitudes to the upper air west wind maximum in the region of latitude 50°, and a smaller indirect "frontal cell" between latitude 30° and 50° which receives angular momentum and energy from the trade wind cell by means of friction. *G. C. McVittie* (Urbana, Ill.).

Montroll, Elliott W., and Hart, Robert W. Scattering of plane waves by soft obstacles. II. Scattering by cylinders, spheroids, and disks. J. Appl. Phys. 22, 1278–1289 (1951).

In an earlier paper [same J. 22, 376–386 (1951); these Rev. 13, 94] these authors have considered the scattering of sound waves by spheres whose index of refraction is close to that of the surrounding medium obtaining an approximate solution in closed analytic form. In the present paper the same technique is applied to infinite cylinders. Prolate spheroids and finite cylinders are considered as perturbations on the infinite cylinder, the value of the wave function inside being taken as that which is found in an infinite cylinder whose radius equals that of the minor axis of the prolate spheroid or that of the finite cylinder. Finite disks are treated in a similar way as perturbations on the infinite disk. Both total and differential cross-sections are obtained. *H. Feshbach* (Cambridge, Mass.).

Elasticity, Plasticity

Williams, E. J., and Kloot, N. H. Stress-strain relationship, a mathematical model. Australian J. Appl. Sci. 3, 1–13 (1952).

Uniaxial stress only is considered but the rate of strain is included. The model is a bundle of a large number of unequal strength elastic fibers carrying equal load until they fail. Failure loads are distributed over any chosen range and stress-strain curves resembling those of various materials are obtained. Rate of loading is taken into account by assuming that a fiber which fails sheds its load gradually as a function of time after failure only. *D. C. Drucker.*

Tiffen, R. Uniqueness theorems of two-dimensional elasticity theory. Quart. J. Mech. Appl. Math. 5, 237–252 (1952).

This paper is concerned with elastic material occupying a multiply-connected region and in a state of plane strain or generalised plane stress. For a finite material it is shown that a solution giving specified stresses over all boundaries is effectively unique, i.e., the stresses are uniquely determined throughout the material and the displacements are unique except for rigid body displacements, and that a solution giving specified displacements over all boundaries is unique. The theorems are stated so that they retain their significance if the boundary loading is discontinuous, and the proofs make no appeal to special forms of the complex potential functions. The hitherto neglected problem of material for which the external contour is partly at infinity and partly in the finite part of the plane is considered, and a solution giving required stresses over all boundaries in the finite part of the plane and specified stresses at infinity of

orders higher than $o(r^{-1})$, where r is the distance from the origin, is proved to be effectively unique. The simple requirement of vanishing stresses at infinity is shown to be, in general, insufficient to define a unique elastic problem, whilst on the other hand stresses at infinity which are $o(r^{-1})$ may not be arbitrarily specified. A solution giving specified displacements over all finite boundaries and prescribed finite and infinite displacements at infinity is shown to be unique, but an arbitrary assignment of infinitesimal displacements at infinity is not permissible. The special cases of material extending to infinity in all directions and of material occupying a half-plane are considered in detail. (From the author's summary.) *R. M. Morris* (Cardiff).

Belen'kiĭ, M. Ya. A mixed problem of the theory of elasticity for an infinitely long strip. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 283–292 (1952). (Russian)

The author solves first an auxiliary problem, the stressed state of an infinite strip, with the condition that at the boundary the normal stress is a given function, $f(x)$, and the shearing stress vanishes. Applying analytic functions of a complex variable, utilizing Cauchy integral formulas, and Fourier transformations the author obtains the expressions involving the stress function in the form of a system of integral equations. The mixed problem involves the following additional boundary conditions: On the segment L which is a part of the boundary, the y-component of the displacement is a given function $\varphi(x)$; the normal stress outside the segment L is a function $\Theta(x)$; the shearing stress vanishes on the entire boundary. Proceeding in the same way as in the auxiliary problem and using the results from the auxiliary problem, again a system of integral equations is obtained. A few examples are worked out where the author demonstrates the approximate methods of solving the above mentioned systems, and also shows how to estimate the accuracy of the approximation. *T. Leser* (Lexington, Ky.).

Timpe, A. Brückenlösungen beim Problem der achsensymmetrischen Torsion. Z. Angew. Math. Mech. **32**, 226–227 (1952).

On the basis of biharmonic solutions given in polar spherical coordinates the author supplements previous solutions of axially symmetric torsion problems [same Z. **30**, 50–61 (1950); these Rev. **11**, 557] with seven additional types. Two illustrative examples are appended showing how new solutions of these axially symmetric cases are obtained by spherical inversion. *D. L. Holl* (Ames, Iowa).

Robinson, Kenneth. Elastic energy of an ellipsoidal inclusion in an infinite solid. J. Appl. Phys. **22**, 1045–1054 (1951).

An exact solution in closed form is presented for the displacements, stresses and total elastic energy for the problem of an ellipsoidal isotropic inclusion bounded to an isotropic body of infinite extent when the inclusion and the infinite solid have uniformly distributed changes in their stress-free specific volumes. The physically important details of the dependence of the total strain-energy of the two materials upon the shape and volume of the inclusion, the elastic moduli of the two isotropic materials and the changes of specific volume, are included. The solution is indicated for the stresses and displacements for the case where the material at infinity is, in addition to the above condition, in a homogeneous state of stress whose principal directions are parallel to the axes of the cavity. (From the author's summary.) *F. Niordson* (Stockholm).

Kovalenko, A. D. The stressed state of a rotating conical shell with wall thickness varying according to a linear law. Akad. Nauk SSSR. Inženernyĭ Sbornik **9**, 143–166 (1951). (Russian)

The author considers a very flat hollow cone whose angle at the vertex is only a little less than 180°. Its thickness varies linearly according to the formula, $h=h_0(1-l/l_0)$, where h_0 is the thickness at the vertex, l_0 is the total length of an element at the middle surface, measured from the vertex to the point where the thickness h is zero, l is the variable length of an element corresponding to the variable thickness h. The wheels of turbo-compressors and of some other machines have often a very similar shape. An identical problem was solved before by E. Honegger. The author of this paper admits that his solution is somewhat similar, but claims that it is much simpler.

The two partial differential equations for the displacements and the stresses are taken from the author's previous (unavailable) work [The theory of the resistance of a turbine wheel, Akad. Nauk Ukrain. SSR, Kiev, 1950]. The author changes these equations into hypergeometric equations using a method referred to V. V. Golubev [Lectures on the analytic theory of differential equations (in Russian), Gostehizdat, Moscow-Leningrad, 1941]. The formulas for stresses, displacements and bending moments found from the solutions of the above mentioned differential equations are given conveniently in terms of certain coefficients and hypergeometric functions. Numerical values of the coefficients for different values of χ (χ is a dimensionless number combining the ratio of the dimensions and the angle at the vertex, $\chi=l_0\cot\alpha/h_0$) are tabulated. The graphs of the stresses and the bending moments plotted against l illustrate well the stressed state of the shell.

The solution of the problem shows that: (1) a rotating hollow conical shell is less resistant than a disc of corresponding dimensions. Even for small values of χ, which characterizes the influence of the conical shape, the maximum normal stresses increase rapidly; (2) in a shell with a circular central hole the stresses at the inner edges increase 1.5–2 times; (3) the outer edges of a conical shell rotating in a field of central force are quite considerably bent. Therefore in the design of a turbine the space between a conical wheel and the stationary parts should be carefully computed. The bending of the outer edge increases when χ increases up to 1.5–2, and then decreases again. *T. Leser.*

Prentis, J. M. On the compression of a cube between rigid rough plates. Quart. J. Mech. Appl. Math. **5**, 253–256 (1952).

The Prager-Synge upper and lower bound technique applied by Edelman to obtain the apparent modulus of elasticity for the circular cylinder is here employed for the cube. The true Young's modulus is found to be 94.2 percent of the apparent within ±0.1 percent. *D. C. Drucker.*

Hemp, W. S. On a theory of sandwich construction. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2672** (**11,568**), 9 pp. (1952).

"The theory of sandwich construction developed in this paper proceeds from the simple assumption that the filling has only transverse direct and shear stiffnesses, corresponding to its functional requirements. This supposition permits integration of the equilibrium equations for the filling. The resulting integrals are used to study the compression buckling of a flat sandwich plate. A second approach to sandwich problems is made in section 5, where a theory of

'bending' of plates is outlined. This generalises the usual theory, making allowance for flexibility in shear." The paper suffers from a lack of references to related literature.

E. Reissner (Cambridge, Mass.).

Rahmatulin, H. A. Normal impact on a flexible cord by a body of given form. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 23–34 (1 plate) (1952). (Russian)

A body of arbitrary form moves with uniform velocity perpendicular to a stretched cord of infinite length. A part of the cord is in contact with the moving body, the rest of the cord is free. The action of the body on the cord is represented by concentrated forces in the points where the cord leaves the body. For a wedge-shaped and a cylindrical body and elastic behavior of the cord, the deformations of the cord are calculated and an extension is given for plastic deformations. For a sharp wedge and small velocity, only the point of the wedge is in contact with the cord, but when the wedge is blunt or the velocity greater, or the body is cylindrical, a finite length of the cord is in contact with the body.

W. H. Muller (Amsterdam).

Oniashvili, O. D. A contribution to the question of stability of a cylindrical shell under axial compression. Soobščeniya Akad. Nauk Gruzin. SSR. **8**, 141–150 (1947). (Russian. English summary)

V. Z. Vlasov [Akad. Nauk SSSR. Prikl. Mat. Meh. **8**, 109–140 (1944); these Rev. **7**, 42] revised the Kirchoff-Love theory of shells, putting the differential equations in a symmetric form. Vlasov's theory is very general and theoretically can be applied to a shell of any form. The author of this paper applies Vlasov's theory to a cylindrical shell. Utilizing also the Galerkin method [ibid. **7**, 49–56 (1943); these Rev. **5**, 139] the author obtains an eighth-order partial differential equation, the unknown function being the stress function. The equations of elastic stability of the shell are obtained from the above mentioned equation by variational methods. The equation is tested first on a ring compressed radially by a constant distributed load. The formula obtained for the critical load agrees with the well-known result derived otherwise. Then the author solves the problem as described in the title, i.e., an axially compressed cylindrical shell of infinite length and then of finite length. In each case symmetrical and non-symmetrical buckling are treated separately.

T. Leser (Lexington, Ky.).

Oniašvili, O. D. On the computation of the critical forces for certain cases of buckling of cylindrical shells. Soobščeniya Akad. Nauk Gruzin. SSR. **8**, 227–234 (1947). (Russian)

In the paper reviewed above the author applied Vlasov's theory of elastic shells to the case of a cylindrical shell, derived for it a stability differential equation, and studied the problem of stability under axial compression. In this paper the author uses his stability equation and continues to study a cylindrical shell examining the critical loads for the following cases: a lateral radially directed distributed load, a combination of lateral and axial loads, and twisting moments applied at the ends.

T. Leser.

Saibel, Edward. Buckling of continuous beams on elastic supports. J. Franklin Inst. **253**, 563–566 (1952).

The problem dealt with is the determination of the stiffnesses of uniform intermediate elastic supports equally spaced along a continuous beam, such that the supports behave as though rigid when axial compression of the beam causes buckling. The author extends an energy argument of Timoshenko [Theory of elastic stability, 1st ed., McGraw-Hill, New York, 1936, p. 106] and shows that the problem is analogous to that of vibration of a stretched cord carrying a set of equal masses equally spaced. Available solutions for this vibration problem afford a simple closed solution for the buckling problem.

M. Goland.

Lembcke, H.-R. Biege- und Torsionsschwingungen von Stäben mit beliebigen Querschnitten. Ing.-Arch. **20**, 91–105 (1952).

The author studies two-fold and three-fold coupled vibrations of beams with variable sections where the elastic axis does not coincide with the centroidal axis as in the case of angle sections with unequal legs. The coupling is for torsional vibration and two-directional bending vibrations. In three-fold coupling Ritz's method is used to obtain the frequency equation. These results for an angle section are compared with the experimental results and good agreement is obtained. The two factors which effect the coupling most markedly are the length of the beam and the distance between the elastic and centroidal axes. In short beams the coupling is more marked while in long beams with suitable end conditions the coupled vibrations differ to a smaller degree from the corresponding free vibrations.

D. L. Holl (Ames, Iowa).

Sveklo, V. A. On a diffraction problem. Doklady Akad. Nauk SSSR (N.S.) **83**, 51–54 (1952). (Russian)

Due semipiani materiali di diverse proprietà elastiche confinano lungo l'asse x. Per $x<0$ essi sono a contatto, mentre per $x>0$ sono separati da una fenditura. Un'onda piana che arriva da uno dei due semipiani dà luogo ad un'onda riflessa, ad un'onda rifratta e a un fenomeno di diffrazione. La trattazione matematica viene ricondotta ad un problema di Dirichlet per un semipiano.

G. Toraldo di Francia (Rochester, N. Y.).

Gutin, L. Ya. On the theory of steady vibrations of an elastic semi-space. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **21**, 892–906 (1951). (Russian)

Il s'agit de calcul des déplacements produits à l'intérieur d'un demi-espace élastique par les ondes superficielles de Rayleigh. Lamb [Philos. Trans. Roy. Soc. London. Ser. A. **203**, 1–42 (1904)] a calculé ces déplacements sur la surface dans le cas des sources de perturbations particulièrement simples. Sobolev et Smirnov [Akad. Nauk SSSR. Trudy Seismol. Inst. No. **20** (1932)], Naryškina [ibid. No. **21** (1933), **45**, **48** (1934), **90** (1940)], et Šerman ont étudié le problème dans des hypothèses beaucoup plus larges sans arriver à des formules asymptotiques faciles à manier. L'auteur montre qu'en utilisant les expressions asymptotiques de déplacements superficiels de Lamb ainsi que les lois de réflexion d'ondes élastiques planes et le principe de réciprocité on peut obtenir des expressions asymptotiques de déplacements intérieurs dûs aux ondes de Rayleigh, ceci à condition de prendre comme sources de perturbations sources simples harmoniques, forces verticales et horizontales, couples de forces, centres de pression, etc. Le mémoire ne contient pas de résultats théoriques nouveaux en comparaison avec les prédécesseurs, mais il a d'avantage de simplicité.

V. A. Kostitzin (Paris).

Jardetzky, W. S., and Press, Frank. Rayleigh-wave coupling to atmospheric compression waves. Bull. Seismol. Soc. America **42**, 135–144 (1952).

Les auteurs obtiennent la courbe de dispersion des ondes de Rayleigh dans le milieu composite suivant: atmosphère,

couche élastique pour laquelle la vitesse des ondes de Rayleigh serait moindre que la vitesse du son dans l'air, milieu élastique où les vitesses seraient grandes. Les calculs numériques montrent que la présence de l'atmosphère crée une nouvelle branche de la courbe de dispersion et qu'elle correspond à des ondes de fréquence à peu près constante. Il est probable que ces ondes ont des amplitudes importantes lorsque une onde aérienne est enregistrée par un séismographe, ou lorsque une onde émise dans le sol est enregistrée par un microphone. *J. Coulomb* (Paris).

Satô, Yasuo. Study on surface waves. V. Love-waves propagated upon heterogeneous medium. Bull. Earthquake Res. Inst. Tokyo **30**, 1–11 (1952). (English. Japanese summary)

L'amplitude des ondes de Love dans un milieu de densité constante mais où la rigidité est fonction linéaire de la profondeur dépend d'une fonction hypergéométrique confluente de Whittaker. Calculs numériques dans un cas particulier où deux couches homogènes surmontent le milieu précédent. *J. Coulomb* (Paris).

Thomas, T. Y. On the characteristic surfaces of the von Mises plasticity equations. J. Rational Mech. Anal. **1**, 343–357 (1952).

It has been shown [P. S. Symonds, Quart. Appl. Math. 6, 448–452 (1949); these Rev. 10, 759] that no real characteristic directions exist in the axial symmetric plasticity problem under the Mises yield condition. The author of the present paper shows that in the case of the general three-dimensional problem under the Mises yield condition, two characteristic surface elements will exist at each point if the second principle strain vanishes (then the axial symmetric case reduces to the known plane strain case). These two elements coincide with the surface elements of maximum and minimum shearing stress. Similarly, it is shown that for the Tresca yield condition, there always exist two characteristic surface elements at each point. Finally, the author proves that for the above yield conditions, the surface elements cannot be extended into families of characteristic surfaces. To prove these results, the author uses tensorial methods. By eliminating the over-stress, a system of four equations in four unknowns (the mean stress and the components of the velocity vector) is obtained. The characteristic manifolds are introduced as those manifolds along which the second derivatives of the velocity and the first derivatives of the mean stress in the normal direction are indeterminate. The resulting algebraic conditions on the strain tensor are formulated in terms of invariant theory and analyzed by introducing geodesic coordinates.

N. Coburn (Ann Arbor, Mich.).

Seth, B. R. Finite elastic-plastic torsion. J. Math. Physics **31**, 84–90 (1952).

Torsion of a solid circular cylinder is treated for large strains using both deformation and flow theories of plasticity. Calling α the angle of twist per unit length, deformation theory is said to lead to the result that the radial normal stress in the elastic region is of order α^2 and in the plastic region of order α. Continuity of the stress is therefore not satisfied. This difficulty does not arise in flow theory. To the second order in α, the torsional couple and the radius of the elastic-plastic boundary are the same as for small strain theory.

D. C. Drucker (Providence, R. I.).

Gaydon, F. A. On the combined torsion and tension of a partly plastic circular cylinder. Quart. J. Mech. Appl. Math. **5**, 29–41 (1952).

Several problems of combined tension and torsion are considered for an incompressible elastic-ideally plastic material obeying the Prandtl-Reuss stress-strain relation: 1) constant ratio of rate of extension to rate of twist; 2) constant twist and increasing extension; 3) fixed extension and increasing twist; 4) constant ratio of load to torque. These are solved numerically. It is shown that the stresses are very close to their asymptotic values when plastic strains are still of the order of elastic strains. *D. C. Drucker.*

Kuvšinskiĭ, E. V. The equations of motion of an incompressible elastic-viscous medium. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **21**, 88–92 (1951). (Russian)

It is shown that the results obtained by Frenkel [Kinetic theory of liquids, Oxford, 1946; these Rev. 9, 168] and Tyabin [same Žurnal 19, 559–560 (1949); these Rev. 11, 220] in their solutions of motion of an elastic-viscous incompressible medium are incorrect due to an erroneous assumption of a commutative property of certain operators. The present author first introduces assumptions regarding the relative magnitudes of velocity and stress gradients and thus obtains operators possessing this property. He then discusses the special cases of quasi-stationary, non-stationary, and purely oscillating motion of the medium. It is shown that in regions of small stresses the elastic properties of the medium are important only in non-stationary processes. For stationary processes, with the exception of certain special cases, the elastic-plastic medium moves essentially as a viscous liquid. *H. I. Ansoff* (Santa Monica, Calif.).

Gerstenkorn, Horst. Über elastische Wellen in kubischen Gittern. Ann. Physik (6) **10**, 80–93 (1952).

Nach der Quantenmechanik wird der elektrische Widerstand bloss von den Wärmeschwingungen der Gitterbausteine und von Gitterfehler verursacht. (Ein ideales Gitter streut eine ebene Elektronenwelle überhaupt nicht.) Ziel der vorliegenden Untersuchung ist deshalb im Zusammenhang mit diesem Problem die in einem kubisch raumzentrierten Gitter auftretenden elastischen Wellen zu berechnen. (Für ein einfaches kubisches Gitter haben dieses Problem Born und v. Kármán behandelt.) Für die bei der Verrückung eines Atoms auftretende potentielle Energie wird der Ansatz

$$V = V_0 + \tfrac{1}{2}\sum_{i,k}\sum_{m,n} V_{mn}^{ik} s_m^i s_n^k$$

benützt, wo $i, k = x, y, z$ ist und die s die Verrückungen der mit m bzw. n numerierten Atome bedeuten. Im betrachteten Gittertyp werden bloss die mit den 8+6 unmittelbaren Nachbaren auftretenden Wechselwirkungen berücksichtigt. Aus Symmetriebetrachtungen folgt dann, dass sich alle V_{mn}^{ik} auf vier Konstanten ($\kappa_1, \kappa_2, \kappa_3$, und κ_4) reduzieren. Mit Hilfe dieser Vereinfachung erhält man dann für den Zusammenhang zwischen Ausbreitungsgeschwindigkeit, Frequenz und Polarisation eine kubische Gleichung, die angenähert gelöst wird. Die Resultate sind besonders im Grenzfall von langen Wellen übersichtlich. Die mit κ bezeichneten Grössen werden mit die (Voigtschen) elastischen Konstanten eines regulären Kristalls (c_{11}, c_{12}, und c_{44}) in Zusammenhang gebracht. Die Grösse

$$\beta_2 = (-c_{11} + c_{12} + 2c_{44})/c_{44}$$

scheint für Metalle, welche in die selbe Gruppe des periodischen Systems gehören, (angenähert) den gleichen Wert zu besitzen. *Th. Neugebauer* (Budapest).

MATHEMATICAL PHYSICS

*Samuel, Herbert L. **Essay in physics.** Harcourt, Brace and Co., New York, N. Y., 1952. x+178 pp. $3.00.

After a general philosophical introduction, the greater part of this book is devoted to a new physical idea: the introduction of a modified ether concept. The ether proposed by the author is a two-state ether. It is a quiescent ether which becomes an activated ether when material bodies move through it. Yet this is not the old ether concept. The author writes: "We mark once more the fundamental difference between the earlier theories of an ether and that now presented. With them the ether would necessarily obstruct the movement of the stars and planets: here the ether is the very cause of the movement itself." (p. 85).

At the end of the book is a letter from Einstein in which he takes exception to some of the author's views: "Therefore, I think, it is not justified to regard mass as something 'real', the field, however, as merely a 'fancy'. These are prejudices that spring from a primitive understanding of the concept of 'reality'." (p. 161).

The arguments of the author are of a purely qualitative character. *L. Infeld* (Warsaw).

Ott, K. Zahlen- oder Grössengleichung? Elemente der Math. 7, 80–85 (1952).

O. Hölder's axioms for "quantity" [Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. 53, 1–64 (1901)] are paraphrased, so as to give an axiom system applicable to "tensors" of various physical dimensions. In particular, "dimension" is defined abstractly. The paraphrase is not simple conceptually. *G. Birkhoff* (Cambridge, Mass.).

Braun, Irit, and Reiner, Markus. Note on dimensions in tensor analysis. Bull. Res. Council Israel 1, no. 4, 81–82 (1952).

In this brief note, the authors consider the problem of assigning dimensions to physical quantities. Three methods are suggested: (1) the dimension of the coordinate variables is zero; (2) the dimension of the coordinate variable is one; (3) each coordinate variable carries the dimension zero or one according to whether the variable is a length or an angle, respectively. By considering the expression for the strain tensor given by M. Brillouin [Les tenseurs . . . , Masson, Paris, 1938], the authors show that only the second method of defining dimensions is physically significant. [Reviewer's note. This problem is discussed in detail in Schouten's new text, Tensor analysis for physicists, Oxford, 1951, Chap. VI; these Rev. 13, 493. The method used by the above authors corresponds to Schouten's "relative" dimension or use of the orthogonal components.] *N. Coburn.*

Jordan, Pascual. Über die Erhaltungssätze der Physik. Z. Naturforschung 7a, 78–81 (1952).

The author states reasons for expecting physical theories to be invariant under a group of transformations in a six-dimensional real space and proposes a particular group of transformations as this underlying group. He does not discuss the connection of this group with the various six-dimensional conformal geometries proposed as the underlying geometries for physical theories. *A. H. Taub.*

Poincelot, Paul. Sur la notion de vitesse de groupe. C. R. Acad. Sci. Paris 234, 2426–2427 (1952).

Optics, Electromagnetic Theory

Kiuti, Masazo. Aberration of decentred optical systems. Sci. Papers Coll. Gen. Ed. Univ. Tokyo 1, 19–36 (1951).

The effect of decentering on the third order (Seidel) image errors is considered in a manner somewhat different from Maréchal's treatment of this subject. Decentering is represented by the lateral coordinates of the centers of curvature of component refracting surfaces. In the course of the analysis the effect of decentering of certain surfaces and groups of surfaces can be examined. The results obtained consist of modified formulae for the Seidel aberrations, these now being dependent also on the decentering data. *E. W. Marchand* (Rochester, N. Y.).

Steel, W. H. Calcul de la répartition de la lumière dans l'image d'une ligne. Rev. Optique 31, 334–340 (1952).

The image of an incoherent light source in the form of a line is studied with respect to the light distribution (a) in the plane of the image and (b) along the optical axis. In each case comparison is made between the direct calculation and that using the Fourier transform, the latter affording a considerable simplification. *E. W. Marchand.*

Tedone, Giuseppe. Sulla configurazione dei fronti d'onda epicentrali in certi mezzi isotropi eterogenei. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 8, 246–253 (1949).

Special forms of the wavefront are considered in cases where the index of refraction is given by a function with central symmetry. Some properties of the wavefront and of its orthogonal trajectories are discussed for this special case, including the meridional radius of curvature. The author seems to be unfamiliar with the recent literature of the subject which contains many of his results [cf., e.g., Z. Instrumentenkunde 53, 436–443 (1933)]. *M. Herzberger* (Rochester, N. Y.).

Maecker, H. Die Grenze der Totalreflexion. I. Strahlenoptische Näherung mit der Wolterschen Strahldefinition. Ann. Physik (6) 10, 115–128 (1952).

L'autore si basa sulla definizione di raggio data da H. Wolter [Z. Naturforschung 5a, 143–153 (1950)] mediante l'intersezione di due superfici d'onda inclinate l'una rispetto all'altra di un angolo infinitesimo. I raggi vengono fatti riflettere secondo le formule di Fresnel. Nella riflessione totale i raggi subiscono anche uno spostamento. L'immagine di un punto luminoso, ottenuta per riflessione totale, non è un punto, ma un anello sfumato. *G. Toraldo di Francia.*

Maecker, H. Die Grenze der Totalreflexion. II. Strenge wellenoptische Berechnung. , Ann. Physik (6) 10, 153–160 (1952).

Il problema studiato nel lavoro recensito sopra viene affrontato con la teoria ondulatoria rigorosa, in un caso particolare. Lo studio semigeometrico faceva risultare una lacuna fra i raggi riflessi parzialmente e quelli riflessi totalmente. È naturale che la trattazione ondulatoria faccia sparire tale lacuna e ponga al suo posto un graduale passaggio. Il metodo seguito è quello classico di scomporre un'onda sferica dipolare in una stella di onde piane, che vengono riflesse secondo le formule di Fresnel. Sostanzial-

mente il problema matematico consiste nello scegliere un opportuno cammino d'integrazione per l'integrale che ne risulta. *G. Toraldo di Francia* (Rochester, N. Y.).

Štepa, N. I. Errors of the third order of cylindrical electron lenses. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **22**, 216–226 (1952). (Russian)

La lente cilindrica considerata è la più generale, risultante da un campo elettrico e da un campo magnetico sovrapposti. Le aberrazioni del terzo ordine vengono calcolate con un metodo di approssimazioni successive, partendo dalle equazioni differenziali di Eulero-Lagrange per le traiettorie elettroniche. I coefficienti delle aberrazioni hanno espressioni molto ingombranti. Oltre alle classiche aberrazioni: sferica, coma, curvatura di campo e distorsione, risultano sei aberrazioni, dovute alla componente della velocità iniziale degli elettroni lungo l'asse del cilindro.

G. Toraldo di Francia (Rochester, N. Y.).

Glazer, V. [Glaser, W.], and Gryumm, G. [Grümm, H.]. Caustic surfaces of electron lenses. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **21**, 1412–1426 (1951). (Russian)

Il calcolo vienne effettuato nell'approssimazione di Seidel. Viene scritta l'equazione delle traiettorie elettroniche nello spazio immagine con coefficienti di Seidel generici. L'equazione della caustica, come luogo d'incontro di traiettorie infinitamente vicine, viene ottenuta sotto forma parametrica. Naturalmente, nel caso che siano assenti i coefficienti anisotropi, la forma della caustica coincide con quella già nota nell'ottica geometrica. La superficie caustica viene effettivamente costruita e fotografata in un caso numerico particolare. *G. Toraldo di Francia* (Rochester, N. Y.).

Glazer, V. [Glaser, W.], and Gryumm, G. [Grümm, H.]. On the wave-mechanical determination of the relations for the intensity in space of the images of electron lenses. I. Determination of the electron intensity for an image given according to geometrical optics. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **21**, 1427–1443 (1951). (Russian)

Gli autori partono dall'equazione di Schrödinger per un elettrone immerso in un campo elettrico e magnetico ed applicano il classico metodo di Sommerfeld e Runge per trovare una soluzione di prime approssimazione (W.K.B.). Discutono le condizioni per la validità della ottica geometrica, trovando le limitazioni

$$|F| \ll 3 \cdot 10^{10} U^{3/2}$$

per la forza elettrica F (volt/metro) e per l'induzione magnetica $|B| \ll 5 \cdot 10^{8} U$ (gauss); U è il potenziale acceleratore. Dalla trattazione geometrica si ritorna a quella ondulatoria nello spazio immagine mediante la scomposizione secondo Debye del fascio elettronico in una stella di onde piane. *G. Toraldo di Francia* (Rochester, N. Y.).

Kel'man, V. M., and Rodnikova, I. V. The motion of charged particles in the magnetic field of a linear current and the electric field of a cylindrical condenser. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **21**, 1364–1369 (1951). (Russian)

Viene ricondotto alle quadrature il problema del moto di particelle cariche nel campo elettrico di un condensatore cilindrico e nel campo magnetico di una corrente lineare continua sull'asse di esso. Scegliendo opportunamente i parametri, tale sistema possiede proprietà di focheggiamento per fasci di particelle cariche.

G. Toraldo di Francia (Rochester, N. Y.).

Agostinelli, Cataldo. Vibrazioni elettromagnetiche in una cavità riempita di dielettrico eterogeneo. Univ. e Politecnico Torino. Rend. Sem. Mat. **10**, 175–210 (1951).

The author first discusses in general terms the theory of electromagnetic oscillations inside a cavity filled with a nonhomogeneous dielectric. He assumes a triply orthogonal coordinate system, q_1, q_2, q_3, with metric $ds^2 = \sum Q_i^2 dq_i^2$, and a perfectly conducting cavity boundary defined by $q_1 = q_1^{(0)}$. If the cavity has an axis of symmetry, it is shown that there exist fields for which the only nonvanishing field components are E_2, H_1, H_3 (magnetic type) or H_2, E_1, E_3 (electric type) respectively. More generally, there can exist fields in which only one component (H_1 or E_1) vanishes if ϵ is a function of q_1 only and if certain conditions are imposed on the geometrical shape of the cavity. For these vibrations the field components are expressed in terms of one scalar potential. The general theory is then applied to a spherical cavity in which ϵ varies with the radius. Upper and lower limits for the resonant frequencies are obtained in terms of the Bessel function zeros appropriate to the problem for constant ϵ, and the author also develops an integral equation of Volterra type for the radial factor in the scalar potential. Finally he obtains a perturbation formula for the change in resonant frequency due to small changes in the dielectric constant.

M. C. Gray (Murray Hill, N. J.).

Zeuli, Tino. Vibrazioni elettromagnetiche in una cavità cilindrica circolare retta riempita di dielettrico eterogeneo, con un involucro perfettamente conduttore. Univ. e Politecnico Torino. Rend. Sem. Mat. **10**, 285–304 (1951).

This is an application of the theory in Agostinelli's paper, reviewed above, to a cylindrical cavity of circular cross-section, defined by $0 \leqq \rho \leqq a$, $0 \leqq z \leqq l$. It is shown that oscillations of electric type may exist inside the cavity either when $H_z = 0$ and $\epsilon = \epsilon(z)$ or when $H_\rho = 0$ and $\epsilon = \epsilon(\rho)$. In the latter case the scalar potential must be independent of the angular coordinate. Similar conditions hold for oscillations of magnetic type. Each of the four types is analyzed in detail, and upper and lower bounds obtained for the resonant frequencies. *M. C. Gray* (Murray Hill, N. J.).

Graffi, Dario. Le guide d'onda. Rend. Sem. Mat. Fis. Milano **21** (1950), 14–27 (1951).

This is an elementary exposition of the theory of propagation of electromagnetic waves in a cylindrical wave guide of rectangular cross-section. It includes a discussion of Brillouin's theory in which the waves in the guide are expressed as sums of plane waves inclined to the guide walls and undergoing internal reflections. *M. C. Gray.*

Graffi, Dario. Un teorema di unicità per le equazioni di Maxwell e sue applicazioni alla teoria delle guide d'onda. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **8** (1950–51), 213–218 (1952).

The uniqueness theorem for Maxwell's equations is enunciated as follows: A harmonic electromagnetic field is uniquely determined in a domain (D) (filled by a medium which may be conducting), bounded in part by a perfect conductor and elsewhere by plane surfaces, separated from each other and simply connected, on which the normal components of the field are assigned. If any or all of the plane surfaces are multiply connected, the theorem will be valid only if the circuitation of the magnetic field is also assigned along the lines which bound these surfaces internally. The theorem is applied to a section of perfectly conducting cylindrical waveguide between two transverse planes, $z = 0$,

8. If the normal field components E_z, H_z are assigned on these two planes, the field inside the cavity is uniquely expressed as sums of TE and TM waves, as long as the planes are simply connected. On the other hand, if the end planes are multiply connected, as in a coaxial cavity, the theorem postulates, in addition, the existence of two TEM waves traveling in opposite directions. *M. C. Gray.*

Heading, J., and Whipple, R. T. P. The oblique reflexion of long wireless waves from the ionosphere at places where the earth's magnetic field is regarded as vertical. Philos. Trans. Roy. Soc. London. Ser. A. **244**, 469–503 (1952).

Wilkes [Proc. Roy. Soc. London. Ser. A. **189**, 130–147 (1947); these Rev. **8**, 614] has derived the relevant equations for the case of a plane wave incident on the ionosphere. With the assumptions that the collision frequency is constant and that the electron density varies exponentially with height the author shows how to solve Wilkes' equations to obtain the reflection and transmission coefficients of the ionosphere. The solution requires a discussion of the following fourth-order differential equation:

$$(*) \qquad (\vartheta-p_1)(\vartheta-p_2)(\vartheta-p_3)(\vartheta-p_4)F=wF.$$

Here $\vartheta=wd/dw$ and p_1, p_2, p_3, p_4 are constants. The author shows that the solutions of (*) are essentially the ${}_0F_3$ hypergeometric functions. The asymptotic behavior of these solutions is obtained by representing the hypergeometric functions as Barnes contour integrals, approximating the gamma functions by Stirling's formula and then using the method of steepest descents on the resulting integral.

B. Friedman (New York, N. Y.).

Arzeliès, Henri. Sur une forme matricielle et tensorielle des relations fondamentales de la théorie magnéto-ionique. C. R. Acad. Sci. Paris **234**, 2430–2432 (1952).

A very brief statement of the most general Maxwell equations as they apply to ionospheric propagation.

E. Weber (Brooklyn, N. Y.).

Schumann, W. O. Über die Dämpfung der elektromagnetischen Eigenschwingungen des Systems Erde-Luft-Ionosphäre. Z. Naturforschung **7a**, 250–252 (1952).

The air is considered as a spherical cavity between the assumed conducting earth and the ionosphere. Using previous work [same Z. **7a**, 149–154 (1952)] on the eigenvalues of this system, the author investigates the absorption produced by the finite conductivity of the ionosphere.

B. Friedman (New York, N. Y.).

Berkeš, Branko. Über eine Berechnungsmethode des magnetischen Feldes. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **7**, 36–47 (1952). (German. Serbo-Croatian summary)

The author uses for the magnetic field the volume integral

$$H=\frac{1}{4\pi}\int_V \rho^{-1}\,\mathrm{curl}\,G\cdot dV,$$

where G is the current density and ρ the radius vector from the source point to the point where the field is desired. In the case of homogeneous conductors the only contribution comes from the boundary surface between conductor and surrounding air so that (1) reduces to a surface integral. This is applied to the infinitely long cylinder conductor, to an infinite very thin sheet and to toroids of rectangular and circular cross-sections. In the simple cases the integrals can be evaluated rather readily. *E. Weber.*

Tyabji, S. F. B. The energy momentum tensor in Dirac's new electromagnetic theory. Nature **170**, 116 (1952).

McMillan, Brockway. Introduction to formal realizability theory. I. Bell System Tech. J. **31**, 217–279 (1952).

McMillan, Brockway. Introduction to formal realizability theory. II. Bell System Tech. J. **31**, 541–600 (1952).

An important result of this paper is the characterization of those matrices $Z(p)$, functions of the frequency parameter p, which can be realized as open-circuit impedance matrices of finite passive networks. The solution of this problem was found independently by Y. Oono [J. Math. and Physics **29**, 13–26 (1950); these Rev. **12**, 567] by the present author, and by M. Bayard [Bull. Soc. Française Electriciens (6) **9**, 497–502 (1949)]. This characterization is provided by the following theorem: "Let $Z(p)$ be an n by n symmetric matrix whose matrix elements $Z_{rs}(p)$ are rational functions of p with real coefficients. For each set of real constants $k_1, \cdots, k_n$, the function $\phi(p)=\sum\sum Z_{rs}(p)k_rk_s$ has a non-negative real part whenever p has a positive real part. Then there exists a finite passive network, a $2n$-pole, which has the impedance matrix $Z(p)$. Conversely the impedance matrix (if it exists) of a finite passive $2n$-pole has the above properties". The proof is modeled after that of Brune for the realizability of two-poles. The realization is achieved with the minimum number of reactive elements. Free use is made of ideal transformers. In the usual mathematical treatment of networks open circuit and short circuit terminal pairs introduce troublesome degeneracies. The author overcomes this difficulty by introducing the notion of "linear correspondence" in a vector space as the abstract equivalent of a physical network. *R. H. Duffin.*

Fialkow, Aaron, and Gerst, Irving The transfer function of general two terminal-pair RC networks. Quart. Appl. Math. **10**, 113–127 (1952).

The transfer function $A(p)$ of a network is the ratio of output to input voltages as a function of the complex frequency p. It has the form

$$A(p)=KN/D$$
$$=K(p^n+a_1p^{n-1}+\cdots+a_n)/(p^m+b_1p^{m-1}+\cdots+b_m)$$

where a_i, b_i, and K are real constants. It may be supposed that N and D have no common factors. It is shown that the necessary and sufficient conditions that a real rational function $A(p)$ be the transfer function of a network made up of resistors and capacitors are: (i) The zeros of D are distinct negative numbers; (ii) $m\geqq n$; (iii) the number K satisfies the inequalities $-K_0<K<K_0$, where K_0 is the least of the three quantities $|K_d|$, $|b_m/a_n|$, 1 if $m=n$ and of the first two quantities if $m>n$. If $K_0\neq|K_d|$, then K may actually equal $\pm K_0$. Here K_d is that real value of k of smallest absolute value (if it exists) for which the equation $D-kN=0$ has a positive multiple root. If one input and one output terminal are joined, the following additional conditions are imposed: (iiia) k, K_d, and K are restricted to positive values; (iv) the zeros of N must not be positive real but are otherwise arbitrary. This precise determination of the range of K is an important contribution. The method of proof actually furnishes a constructive procedure for synthesizing a network having a given transfer function. This procedure may

be modified to accommodate any given resistive source or load. *R. J. Duffin* (Pittsburgh, Pa.).

Quantum Mechanics

Fantappiè, Luigi. Caratterizzazione analitica delle grandezze della meccanica quantica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 285–290 (1952).

The author presents conditions on an hermitian operator which he argues are necessary and sufficient in order that it represent a physically real observable. Actually his conditions would appear to require that any operator on spin-space be a scalar in any relativistic situation in which the pertinent representation of the Lorentz group is irreducible. *I. E. Segal* (Chicago, Ill.).

Falk, G., und Marschall, H. Eine Bemerkung zur Auswahl der physikalisch brauchbaren Lösungen der Schrödinger-Gleichung. Z. Physik **131**, 269–272 (1952).

In several examples of solutions of the Schrödinger wave equation for systems with spherical symmetry, the authors show that it is possible to single out the physically acceptable solutions by means of criteria that fit into the framework of quantum mechanics: normalization of the wave function and Hermiticity of the momentum operator. *N. Rosen* (Chapel Hill, N. C.).

Höhler, G. Zur neuen klassischen Theorie des Elektrons von Dirac. Ann. Physik (6) **10**, 196–200 (1952).

The first form of Dirac's new classical electrodynamics [Proc. Roy. Soc. London. Ser. A. **209**, 291–296 (1951); these Rev. **13**, 893] is obtained in two ways: (i) by transition from the Maxwell-Lorentz equations for electrons of mass m and charge e moving in the field of the other electrons by allowing m and e to approach 0 while m/e remains constant; (ii) by letting $\hbar$ approach 0 in the equations $\partial_\nu F_{\mu\nu}=j_\mu$,

$$[(-i\hbar\partial_\nu+eA_\nu)^2+m^2]\varphi=0,$$
$$j_\mu=(ie\hbar/2m)(\varphi^*\partial_\mu\varphi-\varphi\partial_\mu\varphi^*)-e^2/m\varphi^*\varphi A$$

after φ has been approximated by

$$\varphi=(a_0-i\hbar a_1+\cdots)\exp(iS/\hbar)$$

and S is interpreted as generating a gauge transformation. *A. J. Coleman* (Toronto, Ont.).

Vigier, Jean-Pierre. Introduction géométrique de l'onde pilote en théorie unitaire affine. C. R. Acad. Sci. Paris **233**, 1010–1012 (1951).

de Broglie, Louis. Remarque sur la note précédente de M. Vigier. C. R. Acad. Sci. Paris **233**, 1012–1013 (1951).

de Broglie, Louis. Sur la possibilité d'une interprétation causale et objective de la mécanique ondulatoire. C. R. Acad. Sci. Paris **234**, 265–268 (1952).

Régnier, André, Schatzman, Evry, et Vigier, Jean-Pierre. Sur la répartition statistique des mouvements des particules en mécanique quantique. C. R. Acad. Sci. Paris **234**, 410–412 (1952).

Marx, G. Drehimpuls in der Quantentheorie der Wellenfelder. Acta Phys. Acad. Sci. Hungar. **1**, 209–233 (1952). (Russian summary)

A symmetrical energy-momentum tensor is constructed for higher-order Lagrangians. From this an angular momentum tensor is obtained. It is then shown that the angular momentum of the field may be uniquely decomposed into spin and orbital components. *K. M. Case.*

Ôno, Yôro. On the energy-momentum tensor of Bopp-type non-local field. Progress Theoret. Physics **6**, 925–938 (1951).

The energy-momentum tensor for a Bopp-type non-local field is constructed. This is done by considering the characteristic non-local function as a power series in the D'Alembertian acting on the delta function. Taking the functional derivative of the Lagrangian with respect to the metric tensor yields a term of the momentum tensor corresponding to each term in the Lagrangian. It is shown that the resultant series may be summed and the energy-momentum tensor expressed in terms of the characteristic function. This tensor is shown to be symmetric and gauge invariant and tends to the usual tensors in the various limiting cases. *K. M. Case* (Ann Arbor, Mich.).

Hara, Osamu, and Shimazu, Haruo. On Yukawa's theory of non-local field. I. The case of free field. Progress Theoret. Physics **7**, 255–262 (1952).

Fierz [Physical Rev. **78**, 184 (1950); Helvetica Phys. Acta **23**, 412–416 (1950); these Rev. **12**, 67] pointed out that for free non-interacting scalar fields Yukawa's non-local theory [Physical Rev. **77**, 219–226 (1950); **80**, 1047–1052 (1950); these Rev. **11**, 567; **12**, 571] is equivalent to the superposition of local fields of higher spin. The present paper studies this equivalence and concludes that "when the field is quantized the equations of motion and the commutation relations for the non-local field can be derived from those of the local field by a canonical transformation". Further, if the non-local field is irreducible under spatial rotations in the centre of mass system, it corresponds to an irreducible local field of definite spin. However, Yukawa has already pointed out, in the first section of the second of his above cited papers, that this (1, 1)-correspondence of irreducible local and non-local fields does not obtain for free spinor fields and probably fails altogether for interacting fields. The present authors intend to discuss interacting fields in a subsequent paper. *A. J. Coleman.*

Van Hove, Léon. Sur l'opérateur Hamiltonien de deux champs quantifiés en interaction. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **37**, 1055–1072 (1951).

The author treats on a rigorous level the question of whether the usual formal expression for the Hamiltonian of interacting boson and fermion fields can be interpreted as a bona fide operator on a Hilbert space, with a non-trivial domain. His results tend to indicate that no such strict operator exists, either for the entire Hamiltonian H or for the interaction Hamiltonian H_I, on the usual Hilbert space S spanned by the stationary states of the non-interacting fields. More specifically, let ϕ_α denote an arbitrary normalized stationary state characterized by definite numbers of bosons, fermions, and anti-fermions, in specified plane wave states, the system being assumed enclosed in a large cube, and let S be the Hilbert space of all stationary states $\phi=\sum c_\alpha\phi_\alpha$ with $\|\phi\|^2=\sum_\alpha|c_\alpha|^2$ finite. Let H^K and H_I^K be the parts of the total and interaction Hamiltonians ob-

tained by cutting off at the momentum level K, so that formally $H=\lim_K H^K$ and $H_I=\lim_K H_I^K$; then H^K and H_I^K are densely (but not everywhere) defined operators on S. The main result is that for every nonzero vector ϕ in S in the domains of all the boson annihilation and creation operators (so that $H^K\phi$ and $H_I^K\phi$ are again vectors in S), $\lim_K \|H^K\phi\|=\lim_K \|H_I^K\phi\|=\infty$. When $\phi=\phi_\alpha$ for some α, this result is almost immediate and had in fact previously been mentioned by Snyder [Physical Rev. 79, 520–525 (1950); these Rev. 12, 226]. The derivation of the present result involves the successive treatment of the following cases: (1) fixed non-quantized fermions; (2) quantized fermions neglecting pair production; (3) the general case, the technique for which is built up from the earlier, and especially first, case.

The inequality $\|H_I^K\phi_\delta\|^2\leqq\rho'^2K^2$ on p. 1064 must be restricted to those δ such that ϕ_δ represents a state with a bounded number of bosons, which is adequate for the proof since the δ in A_δ are such. *I. E. Segal* (Chicago, Ill.).

Van Hove, Léon. Les difficultés de divergences pour un modelle particulier de champ quantifié. Physica 18, 145–159 (1952).

In order to study the source of divergence difficulties in quantum field theory, the exact solution of a neutral scalar field in scalar interaction with infinitely heavy point sources is compared with the "solution" obtained by perturbation methods. It is shown that the exact solutions are orthogonal to the stationary states of the free field so that no linear combination of the latter can approximate the former. This situation does not occur for systems involving only a finite number of degrees of freedom. It suggests that there is no mathematical justification for using the interaction representation and that the occasional successes of renormalization methods are lucky flukes which "can be fully understood only when we have a more exact knowledge of the stationary states of the electrodynamic field."

A. J. Coleman (Toronto, Ont.).

Proca, A. Sur l'espace-temps des particules fondamentales et les espaces spinoriels sous-jacents. Bull. Sci. Roumain 1, 18–24 (1952).

It is suggested that spinors and not vectors are the correct quantities to describe "position". Thus in three dimensions two conjugate two-component spinors might provide a better description of nature than the vector components of position (x, y, z). Introducing a fourth "physical" quantity T puts the spinor "position" in one-to-one correspondence with physically observable quantities. The form of various simple operators expressed in terms of the underlying spinor coordinates is given. It is asserted that the formalism seems to provide a simpler description of "spin" than the conventional one. *K. M. Case* (Ann Arbor, Mich.).

Urban, P., und Schwarzl, F. Die Theorien der Teilchen mit höherem Spin. Acta Physica Austriaca 4, 380–404 (1951).

The formulations of relativistic wave equations for higher spin particles of Dirac [Proc. Roy. Soc. London. Ser. A. 155, 447–459 (1936)], Pauli and Fierz [Helvetica Phys. Acta 12, 297–300 (1939)], and Bhabha [Rev. Modern Physics 17, 200–216 (1945); these Rev. 7, 272] are summarized. No discussion of the difficulties encountered by these theories is given. *K. M. Case.*

Osborn, Richard K. The second-quantized theory of spin-½ particles in the nonrelativistic limit. Physical Rev. (2) 86, 340–347 (1952).

A second quantized analog of the transformation of Foldy and Wouthuysen [same Rev. 78, 29–36 (1950)] is exhibited. It is shown that by a canonical transformation the wave operators for positrons and electrons become linearly independent two-component operators. The Hamiltonian separates into an electron and a positron part in the case of no interaction. It is also shown that there exists a sequence of canonical transformations for spin-½ particles interacting via intermediate fields which reduces the Hamiltonian to nonrelativistic form. *K. M. Case* (Ann Arbor, Mich.).

Shanmugadhasan, S. The dynamical theory of magnetic monopoles. Canadian J. Physics 30, 218–225 (1952).

Dirac's theory of magnetic monopoles [Physical Rev. 74, 817–830 (1948); these Rev. 10, 345] is thrown into a form in which charges and monopoles are treated symmetrically, using two four-potentials corresponding to charges and monopoles respectively. The Fermi form of electrodynamics is then extended to this generalized theory.

H. C. Corben (Genoa).

Shanmugadhasan, S. Spinors in the dynamical theory of spinning particles. Canadian J. Physics 30, 226–234 (1952).

The rotational equations of motion of a particle analogous to a pure gyroscope or a symmetrical top are put into spinor form. The Poisson brackets of the conjugate variables in this formalism are obviously covariant, as opposed to the case, previously developed by the author [same J. 29, 593–612 (1951); these Rev. 13, 710] in which the formalism is expressed in terms of quasi-coordinates. Now, however, the canonical variables are not all independent and it is necessary to use Dirac's generalized Hamiltonian dynamics. Under certain conditions of strong equality, the Poisson brackets developed here agree with those of the other formulation. *H. C. Corben* (Genoa).

Pirenne, Jean. Covariant theory of radiation damping. Physical Rev. (2) 86, 395–398 (1952).

The identity is established between Schwinger's covariant expression for the S-matrix and Pauli's non-covariant expression used in Heitler's theory of radiation damping. From comparison of the two it is pointed out that mass renormalization is needed for internal consistency, quite apart from the problem of divergences. Expressing S in the Cayley form in terms of $\bar{K}$ ($\bar{K}$ Hermitean) the author investigates the relations between the nth order approximations to $\bar{K}$ and S. *H. C. Corben* (Genoa).

Kita, Hideji. Relativistic two-body problem. Progress Theoret. Physics 7, 217–224 (1952).

An inhomogeneous integral equation is obtained for the scattering of two particles by Feynman's method and its properties are examined for the case in which the homogeneous part corresponds to an incident plane wave. A homogeneous integral equation is then derived for the bound states of two particles [cf. Salpeter and Bethe, Physical Rev. 84, 1232–1242 (1951)]. *H. C. Corben* (Genoa).

Halpern, Otto, and Hall, Harvey. Scattering of radiation by electrons in relativistic quantum mechanics. Physical Rev. (2) 84, 997–1008 (1951).

The question of the equivalence of the matrix elements of the Dirac one-particle and many-particle theories is re-

opened, and it is argued that, although this equivalence exists for the matrix elements of the relativistic Compton effect, it does not exist for coherent scattering or incoherent scattering by bound electrons. In the latter case, for example, the many-particle theory yields also the possibility of two electrons and a positron in the intermediate state and it is argued that their mutual interaction should be considered. This could yield important differences for very hard radiation. It would be interesting to see the authors' arguments expressed in terms of the Feynman formalism, in which of course, the different types of intermediate states are included in the one expression. *H. C. Corben.*

Breit, G. Topics in scattering theory. Rev. Modern Physics **23**, 238–252 (1951).

A review of the theory of scattering with emphasis on the fact that the variational method rarely possesses advantages over the direct method, except as a guide to the existence of certain identities. Since these identities depend on the existence, rather than on the actual form, of the variational principle, they may be established without recourse to the variational method. The connection is established between the relation connecting energy change, perturbing potential, and initial and final wave functions, and the Ferretti-Hulthén exact relation between phase-shift change and initial and final wave functions. The function of energy and phase-shift f, introduced by Breit, Condon and Present, is expanded in powers of energy and of parameters related to the assumed form for the potential energy, and the method of the f-function is extended to the many-body case.

H. C. Corben (Genoa).

Goldberger, Marvin L. Approximation methods in the theory of scattering. Physical Rev. (2) **84**, 929–938 (1951).

In this extension of a previous paper [same Rev. **82**, 757 (1951)] there is developed and discussed a generalization of the integral equation of scattering to two other equations, the approximate solution to one of which, when substituted into the other, yields the ordinary Heitler integral equation. These equations are thrown into a variational form. This variational method is then applied to the nonrelativistic Compton effect, meson-nucleon scattering, and the scattering of a scalar meson by a fixed extended source, and a formal solution of the general scattering problem is developed. *H. C. Corben* (Genoa).

Majumdar, R. C., Apte, A. S., and Sundaresan, M. K. The scattering of mesons by nuclear particles. I. Progress Theoret. Physics **6**, 737–747 (1951).

A modification of the Duffin-Kemmer matrix formalism for meson theory which has been proposed by Harish-Chandra [Proc. Roy. Soc. London. Ser. A. **186**, 502–525 (1946); these Rev. **8**, 302] is shown to make possible considerable simplification of the treatment of relativistic meson scattering in radiation damping theory. The usual spur calculations are replaced by repeated use of algebraic relations involving Harish-Chandra's matrices. The calculations are worked out in detail for the scattering of pseudoscalar and vector mesons by nuclear particles; in either case the possibility is taken into consideration that the meson can be pseudoscalar or vector in the final state. The considerations are restricted to charged mesons because the integral equation of the radiation damping theory can be solved exactly only in this case. *E. Gora* (Providence, R. I.).

Yamazaki, Kazuo, and Enatsu, Hiroshi. On the self-energies of mesons. Progress Theoret. Physics **6**, 731–736 (1951).

In two previous papers [Enatsu, same journal **6**, 643–664 (1951); Enatsu and Pac, ibid. **6**, 665–672 (1951); these Rev. **13**, 1012] it has been shown that all the divergencies in the expressions for the self-energies of nucleons can be eliminated by mixing different types of meson fields in a suitable way. The same procedure is now employed to investigate the meson self-energies. It appears that, contrary to the case previously considered, it is not possible to remove all the divergencies just by mixing various Fermion fields.

E. Gora (Providence, R. I.).

Borowitz, Sidney, and Kohn, Walter. On the stress tensor of the electron. Physical Rev. (2) **86**, 985–995 (1952).

The formal vanishing of the divergence of the electron-electromagnetic field stress tensor is used to obtain a zero value for certain ambiguous integrals. This result is then used to show that the self-stress of the electron is zero and that the matrix elements of the symmetric stress tensor between states of unequal momenta are finite. The connection with the results of Villars [same Rev. **79**, 122–128 (1950); these Rev. **12**, 150], Rohrlich [ibid. **77**, 357–360 (1950)], and Pais and Epstein [Rev. Modern Physics **21**, 445–446 (1949)] is discussed. *K. M. Case.*

Fues, E., und Statz, H. Ersatzpotentiale mit verwandtem Eigenwertspektrum in Schrödinger-Gleichungen. Z. Naturforschung **7a**, 2–9 (1952).

The reason why valence electrons in some metals can be treated as a gas of free electrons is investigated. The conditions for which the potential holes of the lattice have only slight influence on the energy spectrum of the valence electrons are derived. One can then replace the potential holes in the Schrödinger equation by auxiliary potentials which are easier to handle but which lead to approximately the same eigenfunctions and eigenvalues. A peculiar behaviour of electrons in some metals is explained by the fact that the auxiliary potentials are nearly constant in these cases.

T. E. Hull (Vancouver, B. C.).

Bates, D. R. Rate of formation of molecules by radiative association. Monthly Not. Roy. Astr. Soc. **111**, 303–314 (1951).

In the theory of formation of molecules by radiative association, one has to consider the following problem: Suppose that two normal atoms with a law of attraction $U(r)$ approach each other from infinity. Further let the probability that, when the two atoms are at a distance r, radiation is emitted (by transition to a lower potential energy curve) and a molecule is formed be $A(r)$. Then the probability of formation of a molecule during a collision is $\int A(r)\,dt$ the integration being carried out for all time t for which r is such that the emission of a photon deprives the system of so much energy that the atoms cannot separate to infinity again. To obtain the rate of molecular formation associated with the process it is necessary to average over all values of the relative velocity v and of the impact parameter p. By the laws of motion dr and dt are connected by the relation $dt=dr[v^2-p^2v^2/r^2-2U(r)/m]^{-1/2}$ where m is the reduced mass. For a Maxwellian distribution of velocities at a temperature T the required coefficient γ is

$$\gamma=8\pi^2\left(\frac{m}{2\pi kT}\right)^{3/2}\int\int\int v^3A(r)p[v^2-p^2v^2/r^2-2U(r)/m]^{-1/2}\times e^{-mv^2/2kT}\,dp\,dr\,dv.$$

The integration over p (to be effected from 0 to a maximum value p_M which is consistent with the assigned values for r and v) can be performed and we are left with

$$(*)\quad \gamma = 8\pi^2\left(\frac{m}{2\pi kT}\right)^{3/2}\int\int vr^2A(r)e^{-mv^2/2kT} \times\left[\left\{v^2-2\frac{U(r)}{m}\right\}^{1/2}-\left\{v^2-\frac{2U(r)}{m}-\frac{p_M^2v^2}{r^2}\right\}^{1/2}\right]dr\,dv.$$

At the perihelion point dr/dt vanishes and in the earlier discussion of this problem, it has been inferred that p_M is "therefore" given by the equation $p_M=[r^2-2U(r)r^2/mv^2]^{1/2}$, in which case the second term in brackets under the integral signs in (*) vanishes. The author points out that this inference is in general invalid: it is valid only if $U(r)$ falls off more slowly than r^{-2} so that $[r^2-2U(r)r^2/mv^2]$ is a monotonically increasing function of r^2. But if $U(r)$ falls off more quickly than r^{-2}, the situation is entirely different: the function first passes through a maximum then through a minimum and finally increases monotonically. It is therefore necessary to allow for this in the evaluation of the integral (*) representing γ. The author does this and corrects the earlier calculations on the subject. *S. Chandrasekhar.*